

Weyl Numbers of Embeddings of Tensor Product Besov Spaces

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Abstract

In this paper we investigate the asymptotic behaviour of Weyl numbers of embeddings of tensor product Besov spaces into Lebesgue spaces. These results will be compared with the known behaviour of entropy numbers.

1 Introduction

Weyl numbers have been introduced by Pietsch [32]. They are relatives of approximation numbers. Recall, the n -th approximation number of the linear operator $T \in \mathcal{L}(X, Y)$ is defined to be

$$a_n(T) := \inf \{ \|T - A\| : A \in \mathcal{L}(X, Y), \text{rank}(A) < n \}, \quad n \in \mathbb{N}. \quad (1.1)$$

Here X and Y are quasi-Banach spaces. Now we are in position to define Weyl numbers. The n -th Weyl number of the linear operator $T \in \mathcal{L}(X, Y)$ is given by

$$x_n(T) := \sup \{ a_n(TA) : A \in \mathcal{L}(\ell_2, X), \|A\| \leq 1 \}, \quad n \in \mathbb{N}.$$

Approximation and Weyl numbers belong to the family of s -numbers, see Section 4 for more details. The particular interest in Weyl numbers stems from the fact that they are the smallest known s -number satisfying the famous Weyl-type inequalities, i.e.,

$$\left(\prod_{k=1}^{2n-1} |\lambda_k(T)| \right)^{1/(2n-1)} \leq \sqrt{2e} \left(\prod_{k=1}^n x_k(T) \right)^{1/n} \quad (1.2)$$

holds for all $n \in \mathbb{N}$, in particular,

$$|\lambda_{2n-1}(T)| \leq \sqrt{2e} \left(\prod_{k=1}^n x_k(T) \right)^{1/n},$$

see Pietsch [32] and Carl, Hinrichs [11]. Here $T : X \rightarrow X$ is a compact linear operator in a Banach space X and $(\lambda_n(T))_{n=1}^\infty$ denotes the sequence of all non-zero eigenvalues of T , repeated according to algebraic multiplicity and ordered such that

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \dots \geq 0.$$

Also as a consequence of (1.2) one obtains, for all $p \in (0, \infty)$, the existence of a constant c_p (independent of T) s.t.

$$\left(\sum_{n=1}^{\infty} |\lambda_n(T)|^p \right)^{1/p} \leq c_p \left(\sum_{n=1}^{\infty} x_n^p(T) \right)^{1/p}.$$

Hence, Weyl numbers may be seen as an appropriate tool to control the eigenvalues of T . Many times operators of interest can be written as a composition of an identity between appropriate function spaces and a further bounded operator, see, e.g., the monographs of König [27] and of Edmunds, Triebel [18]. This motivates the study of Weyl numbers of identity operators. Pietsch [34], Lubitz [29], König [27] and Caetano [8, 9, 10] studied the Weyl numbers of $id : B_{p_1, q_1}^t((0, 1)^d) \rightarrow L_{p_2}((0, 1)^d)$, where $B_{p_1, q_1}^t((0, 1)^d)$ denotes the isotropic Besov spaces. Zhang, Fang and Huang [65] and Gasiiorowska and Skrzypczak [22] investigated the case of embeddings of weighted Besov spaces, defined on \mathbb{R}^d , into Lebesgue spaces. Here we are interested in the investigation of the asymptotic behaviour of Weyl numbers of the identity

$$id : S_{p_1, p_1}^t B((0, 1)^d) \rightarrow L_{p_2}((0, 1)^d),$$

where $S_{p_1, p_1}^t B((0, 1)^d)$ denotes a d -fold tensor product of univariate Besov spaces $B_{p_1, p_1}^t(0, 1)$. This notation is chosen in accordance with the fact that $S_{p_1, p_1}^t B((0, 1)^d)$ can be interpreted as a special case of the scale of Besov spaces of dominating mixed smoothness, see Section 5.

The behaviour of $x_n(id : S_{p_1, p_1}^t B((0, 1)^d) \rightarrow L_{p_2}((0, 1)^d))$, $1 < p_2 < \infty$, will be discussed in Subsection 3.1. Here, up to some limiting situations, we have the complete picture, i.e., we know the exact asymptotic behaviour of the Weyl numbers. For the extreme cases $p_2 = \infty$ and $p_2 = 1$, see Subsection 3.2 and Subsection 3.3, we are also able to describe the behaviour in almost all situations. In Subsection 3.4 we discuss the behaviour of $x_n(id : S_{p_1, p_1}^t B((0, 1)^d) \rightarrow \mathcal{Z}_{\text{mix}}^s((0, 1)^d))$, where $\mathcal{Z}_{\text{mix}}^s((0, 1)^d)$ denotes a space of Hölder-Zygmund type. Finally, in Subsection 3.5, we compare the behaviour of Weyl numbers with that one of entropy numbers.

Summarizing we present an almost complete picture of the behaviour of $x_n(id : S_{p_1, p_1}^t B((0, 1)^d) \rightarrow L_{p_2}((0, 1)^d))$, $0 < p_1 \leq \infty$, $1 \leq p_2 \leq \infty$, $t > \max(0, \frac{1}{p_1} - \frac{1}{p_2})$. This is a little bit surprising since the corresponding results for approximation numbers are much less complete, see, e.g., [5]. Let us mention in this context that Weyl numbers have some specific properties not shared by approximation numbers like interpolation properties, see Theorem 4.2, or the inequality (3.3).

The paper is organized as follows. Our main results are discussed in Section 3. In Section 4 we recall the definition of s -numbers and discuss some further properties of Weyl numbers. Section 5 is devoted to the function spaces under consideration. In Section 6 we investigate the Weyl numbers of embeddings of certain sequence spaces associated to tensor product Besov spaces and spaces of dominating mixed smoothness. This will be followed by Section 7, where, beside others, Theorem 3.1 (our main result) will be proved. In Appendix A we recall the behaviour of the Weyl numbers of embeddings $id_{p_1, p_2}^m : \ell_{p_1}^m \rightarrow \ell_{p_2}^m$. Finally, in Appendix B, a few more facts about the Lizorkin-Triebel spaces of dominating mixed smoothness $S_{p, q}^t F(\mathbb{R}^d)$, $S_{p, q}^t F((0, 1)^d)$ and the Besov spaces of dominating mixed smoothness $S_{p, q}^t B(\mathbb{R}^d)$, $S_{p, q}^t B((0, 1)^d)$ are collected.

Notation

As usual, \mathbb{N} denotes the natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, \mathbb{Z} the integers and \mathbb{R} the real numbers. For a real number a we put $a_+ := \max(a, 0)$. By $[a]$ we denote the integer part of a . If $\bar{j} \in \mathbb{N}_0^d$, i.e., if $\bar{j} = (j_1, \dots, j_d)$, $j_\ell \in \mathbb{N}_0$, $\ell = 1, \dots, d$, then we put

$$|\bar{j}|_1 := j_1 + \dots + j_d.$$

By Ω we denote the unit cube in \mathbb{R}^d , i.e., $\Omega := (0, 1)^d$. If X and Y are two quasi-Banach spaces, then the symbol $X \hookrightarrow Y$ indicates that the embedding is continuous. As usual, the symbol c denotes positive constants which depend only on the fixed parameters t, p, q and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. Sometimes we will use the symbols “ \lesssim ” and “ \gtrsim ” instead of “ \leq ” and “ \geq ”, respectively. The meaning of $A \lesssim B$ is given by: there exists a constant $c > 0$ such that $A \leq cB$. Similarly \gtrsim is defined. The symbol $A \asymp B$ will be used as an abbreviation of $A \lesssim B \lesssim A$. For a discrete set ∇ the symbol $|\nabla|$ denotes the cardinality of this set. Finally, the symbols id, id^* will be used for identity operators, id^* mainly in connection with sequence spaces. The symbol id_{p_1, p_2}^m refers to the identity

$$id_{p_1, p_2}^m : \ell_{p_1}^m \rightarrow \ell_{p_2}^m. \quad (1.3)$$

Tensor products of Besov and Sobolev spaces are investigated in [50], [48], [49] and Hansen [24]. General information about Besov and Lizorkin-Triebel spaces of dominating mixed smoothness can be found, e.g., in [1, 2, 3, 4, 46, 47, 61] ($S_{p, q}^t B(\mathbb{R}^d)$, $S_{p, q}^t F(\mathbb{R}^d)$). The (Fourier analytic) definitions of these spaces are reviewed in the Appendix B. The reader, who is interested in more elementary descriptions of these spaces, e.g., by means of differences, is referred to [1, 47] and [60].

2 Some preparations

As a preparation for the main results we recall under which conditions the identity $S_{p_1, p_1}^t B((0, 1)^d) \hookrightarrow L_{p_2}((0, 1)^d)$ is compact, see Vybiral [61, Thm. 3.17].

Proposition 2.1. *The following assertions are equivalent:*

- (i) *The embedding $S_{p_1, p_1}^t B((0, 1)^d) \hookrightarrow L_{p_2}((0, 1)^d)$ is compact;*
- (ii) *We have*

$$t > \max \left(0, \frac{1}{p_1} - \frac{1}{p_2} \right). \quad (2.1)$$

Since we are exclusively interested in compact embeddings the restriction (2.1) will be always present.

Also for later use, we recall the Weyl numbers of the embedding $id : B_{p_1, q_1}^t(0, 1) \rightarrow L_{p_2}(0, 1)$. Let $0 < p_1, q_1 \leq \infty$, $1 \leq p_2 \leq \infty$ and $t > \max(0, 1/p_1 - 1/p_2)$. Then, in all cases listed in Prop. 2.2, we have

$$x_n(id) = x_n(id : B_{p_1, q_1}^t(0, 1) \rightarrow L_{p_2}(0, 1)) \asymp n^{-\alpha}, \quad n \in \mathbb{N}. \quad (2.2)$$

Here the value of $\alpha = \alpha(t, p_1, p_2)$ is given in the following proposition.

Proposition 2.2. *The value of α in (2.2) is given by*

<i>I</i>	$\alpha = t$	<i>if</i> $p_1, p_2 \leq 2$;
<i>II</i>	$\alpha = t + \frac{1}{p_2} - \frac{1}{2}$	<i>if</i> $p_1 \leq 2 \leq p_2$;
<i>III</i>	$\alpha = t + \frac{1}{p_2} - \frac{1}{p_1}$	<i>if</i> $2 \leq p_1 \leq p_2$;
<i>IV*</i>	$\alpha = t + \frac{1}{p_2} - \frac{1}{p_1}$	<i>if</i> $2 \leq p_2 < p_1$ and $t > \frac{1/p_2 - 1/p_1}{p_1/2 - 1}$;
<i>IV*</i>	$\alpha = \frac{tp_1}{2}$	<i>if</i> $2 \leq p_2 < p_1 < \infty$ and $t < \frac{1/p_2 - 1/p_1}{p_1/2 - 1}$;
<i>V*</i>	$\alpha = t - \frac{1}{p_1} + \frac{1}{2}$	<i>if</i> $p_2 \leq 2 < p_1$ and $t > \frac{1}{p_1}$;
<i>V*</i>	$\alpha = \frac{tp_1}{2}$	<i>if</i> $p_2 \leq 2 < p_1 < \infty$ and $t < \frac{1}{p_1}$.

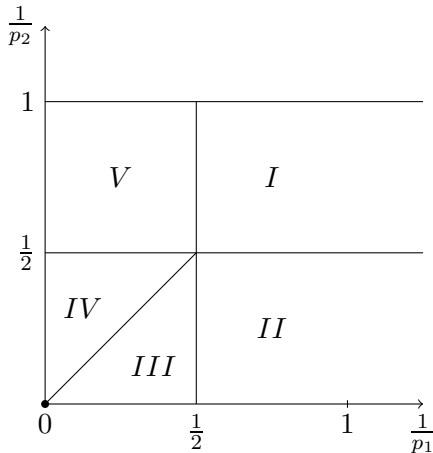


Figure 1

The above results indicate a decomposition of the $(1/p_1, 1/p_2)$ -plane into five parts. In regions *IV* and *V* we have a further splitting into the cases of small (*IV**, *V**) and large smoothness (*IV**, *V**). Proposition 2.2 has been proved by Pietsch [34] and Lubitz [29, Satz 4.13] in case $1 \leq p_1, q_1, p_2 \leq \infty$, we refer also to König [27] in this context.

The proof of the general case, i.e., also for values of p_1, q_1 less than 1, can be found in the

thesis of Caetano [10], see also [9]. Obviously we do not have a dependence on the fine-index q_1 . This will be different in the dominating mixed case with $d > 1$. The behaviour of the Weyl numbers in the limiting situations $t = \frac{1/p_2 - 1/p_1}{p_1/2 - 1}$ (see IV_*, IV^*) and $t = 1/p_1$ (see V_*, V^*) is open, in particular it seems to be unknown whether the behaviour is still polynomial in n .

3 The main results

It seems to be appropriate to split our considerations into the three cases: (i) $1 < p_2 < \infty$, (ii) $p_2 = \infty$ and (iii) $p_2 = 1$.

3.1 The Littlewood-Paley case

Littlewood-Paley analysis is one of the main tools to understand the behaviour of the Weyl numbers if $1 < p_2 < \infty$ (i.e., the target space L_{p_2} allows a Littlewood-Paley-type decomposition). The cases $p_2 = 1$ and $p_2 = \infty$ require different techniques and will be treated in the next subsections. As in the isotropic case the results suggest to work with the same decomposition of the $(1/p_1, 1/p_2)$ -plane as in Proposition 2.2. So, the symbols I, II, \dots , used below, have the same meaning as in Figure 1 (and therefore as in Prop. 2.2). In addition the regions I^* and I_* are given by $p_1, p_2 \leq 2$.

Let $0 < p_1 \leq \infty$, $1 < p_2 < \infty$ and $t > \max(0, 1/p_1 - 1/p_2)$. Then in all cases, listed in Theorem 3.1, we have

$$x_n(id) = x_n(id : S_{p_1, p_1}^t B((0, 1)^d) \rightarrow L_{p_2}((0, 1)^d)) \asymp n^{-\alpha} (\log n)^{(d-1)\beta}, \quad n \geq 2. \quad (3.1)$$

The values of $\alpha = \alpha(t, p_1, p_2)$ and $\beta = \beta(t, p_1, p_2)$ are be given in the following theorem.

Theorem 3.1. *The values of α and β in (3.1) are given by*

$$\begin{aligned} I^* & \quad \alpha = t \text{ and } \beta = t + \frac{1}{2} - \frac{1}{p_1} \quad \text{if} \quad t > \frac{1}{p_1} - \frac{1}{2}; \\ I_* & \quad \alpha = t \text{ and } \beta = 0 \quad \text{if} \quad t < \frac{1}{p_1} - \frac{1}{2}; \\ II & \quad \alpha = t - \frac{1}{2} + \frac{1}{p_2} \text{ and } \beta = t + \frac{1}{p_2} - \frac{1}{p_1}; \\ III & \quad \alpha = t - \frac{1}{p_1} + \frac{1}{p_2} \text{ and } \beta = t + \frac{1}{p_2} - \frac{1}{p_1}; \\ IV^* & \quad \alpha = t - \frac{1}{p_1} + \frac{1}{p_2} \text{ and } \beta = t + \frac{1}{p_2} - \frac{1}{p_1} \quad \text{if} \quad t > \frac{1/p_2 - 1/p_1}{p_1/2 - 1}; \\ IV_* & \quad \alpha = \frac{tp_1}{2} \text{ and } \beta = t + \frac{1}{2} - \frac{1}{p_1} \quad \text{if} \quad t < \frac{1/p_2 - 1/p_1}{p_1/2 - 1}; \\ V^* & \quad \alpha = t - \frac{1}{p_1} + \frac{1}{2} \text{ and } \beta = t + \frac{1}{2} - \frac{1}{p_1} \quad \text{if} \quad t > \frac{1}{p_1}; \\ V_* & \quad \alpha = \frac{tp_1}{2} \text{ and } \beta = t + \frac{1}{2} - \frac{1}{p_1} \quad \text{if} \quad t < \frac{1}{p_1}. \end{aligned}$$

Thm. 3.1 gives the final answer about the behaviour of the x_n in almost all cases. It is interesting to notice that in regions I, IV and V we have a different behaviour for small smoothness compared with large smoothness. Only in the resulting limiting cases we are not able to characterize the behaviour of the $x_n(id)$. However, estimates from below and above also for these limiting situations will be given in Subsection 6.3.

In essence the proof is standard. Concerning the estimate from above the first step consists in a reduction step. By means of wavelet characterizations we switch from the consideration of $id : S_{p_1, p_1}^t B((0, 1)^d) \rightarrow L_{p_2}((0, 1)^d)$ to $id^* : s_{p_1, p_1}^{t, \Omega} b \rightarrow s_{p_2, 2}^{0, \Omega} f$, where $s_{p, q}^{t, \Omega} b$ and $s_{p, q}^{t, \Omega} f$ are appropriate sequence spaces. Next, this identity is splitted into $id^* = \sum_{\mu=0}^{\infty} id_{\mu}^*$ (the id_{μ}^* are identities with respect to certain subspaces) which results in an estimate of $x_n(id^* : s_{p_1, p_1}^{t, \Omega} b \rightarrow s_{p_2, 2}^{0, \Omega} f)$

$$x_n(id^*) \leq \sum_{\mu=0}^J x_{n_{\mu}}(id_{\mu}^*) + \sum_{\mu=J+1}^L x_{n_{\mu}}(id_{\mu}^*) + \sum_{\mu=L+1}^{\infty} \|id_{\mu}^*\|$$

where $n - 1 = \sum_{\mu=0}^L (n_{\mu} - 1)$, see (6.2). Till this point we would call the proof standard, compare, e.g., with Vybiral [62]. But now the problem consists in choosing J, L and n_{μ} in a way leading to the desired result. This is the real problem which we solved in Subsection 6.3. In a further reduction step estimates of $x_{n_{\mu}}(id_{\mu}^*)$ are traced back to estimates of $x_{n_{\mu}}(id_{p_1, p_2}^{D_{\mu}})$, see (1.3). All what is needed about these number is collected in Appendix A. Concerning the estimate from below one has to figure out appropriate subspaces of $S_{p_1, p_1}^t B((0, 1)^d)$ ($s_{p_1, p_1}^{t, \Omega} b$). Then, also in this case, all can be reduced to the known estimates of $x_n(id_{p_1, p_2}^{D_{\mu}})$.

3.2 The extreme case $p_2 = \infty$

Let us recall a result of Temlyakov [53], see also [14].

Proposition 3.2. *Let $t > \frac{1}{2}$. Then we have*

$$x_n(id : S_{2,2}^t B((0, 1)^d) \rightarrow L_{\infty}((0, 1)^d)) \asymp \frac{(\log n)^{(d-1)t}}{n^{t-\frac{1}{2}}}, \quad n \geq 2.$$

Remark 3.3. (i) In the literature many times the notation $H_{\text{mix}}^t((0, 1)^d)$ and $MW_2^t((0, 1)^d)$ are used instead of $S_{2,2}^t B((0, 1)^d)$.

(ii) In [53] and [14] the authors deal with approximation numbers $a_n(id : S_{2,2}^t B((0, 1)^d) \rightarrow L_{\infty}((0, 1)^d))$, see (1.1). However, for Banach spaces Y and Hilbert spaces H we always have

$$x_n(T : H \rightarrow Y) = a_n(T : H \rightarrow Y),$$

see [35, Prop. 2.4.20].

By using specific properties of Weyl numbers we will extent Prop. 3.2 to the following result.

Theorem 3.4. *Let $0 < p_1 \leq \infty$. Then we have*

$$x_n(id : S_{p_1, p_1}^t B((0, 1)^d) \rightarrow L_\infty((0, 1)^d)) \quad (3.2)$$

$$\asymp \begin{cases} \frac{(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_1})}}{n^{t-\frac{1}{2}}} & \text{if } 0 < p_1 \leq 2, t > \frac{1}{p_1}; \\ \frac{(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_1})}}{n^{t-\frac{1}{p_1}}} & \text{if } 2 < p_1 \leq \infty, t > \frac{1}{2} + \frac{1}{p_1}; \end{cases}$$

for all $n \geq 2$.

Remark 3.5. (i) Recall that $S_{p_1, p_1}^t B((0, 1)^d)$ is compactly embedded into $L_\infty((0, 1)^d)$ if and only if $t > 1/p_1$, see Prop. 2.1. The cases $2 < p_1 \leq \infty$ and $t \in (\frac{1}{p_1}, \frac{1}{p_1} + \frac{1}{2}]$ remain open.

(ii) Considering $p_2 \rightarrow \infty$ in parts II and III of Thm. 3.1 then it turns out that in (3.2) there is an additional log factor, more exactly $(\log n)^{(d-1)/2}$.

(iii) To prove Theorem 3.4 we shall employ an inequality due to Pietsch [33]. For any linear operator T we have

$$n^{1/2} x_n(T) \leq \pi_2(T), \quad n \in \mathbb{N}, \quad (3.3)$$

where $\pi_2(T)$ refers to the 2-summing norm of T .

(iv) There is a small number of cases, where the exact order of $s_n(id : S_{p_1, q_1}^t B((0, 1)^d) \rightarrow L_\infty((0, 1)^d))$, $p_1 \neq 2$, if n tends to infinity, has been found. Here s_n stands for any s -number, see Sect. 4. Beside Prop. 3.2 we refer to Temlyakov [54] where

$$d_n(id : S_{\infty, \infty}^t B(\mathbb{T}^2) \rightarrow L_\infty(\mathbb{T}^2)) \asymp n^{-t} (\log n)^{t+1}, \quad t > 0,$$

is proved for all $n \geq 2$. Here d_n denotes the n -th Kolmogorov number and \mathbb{T}^2 refers to the two-dimensional periodic case. For some partial results (i.e., with a gap between the estimates from above and below) with respect to Kolmogorov numbers we refer to Romanyuk [43].

3.3 The extreme case $p_2 = 1$

Let us recall a result obtained by Romanyuk [44] (again Romanyuk has dealt with approximation numbers, but see Remark 3.3 for this).

Proposition 3.6. *Let $t > 0$. Then we have*

$$x_n(id : S_{2, 2}^t B((0, 1)^d) \rightarrow L_1((0, 1)^d)) \asymp \frac{(\log n)^{(d-1)t}}{n^t}, \quad n \geq 2.$$

By making use of the embedding $S_{1, 2}^0 F((0, 1)^d) \hookrightarrow L_1((0, 1)^d)$ we are able to extend Prop. 3.6 to the following.

Theorem 3.7. Let $0 < p_1 \leq \infty$ and $t > (\frac{1}{p_1} - 1)_+$. Then

$$x_n(id : S_{p_1, p_1}^t B((0, 1)^d) \rightarrow L_1((0, 1)^d)) \asymp \begin{cases} n^{-t} & \text{if } p_1 < 2, \quad t < \frac{1}{p_1} - \frac{1}{2}, \\ n^{-t}(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_1})} & \text{if } p_1 \leq 2, \quad t > \frac{1}{p_1} - \frac{1}{2}, \\ n^{-t+\frac{1}{p_1}-\frac{1}{2}}(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_1})} & \text{if } 2 < p_1 \leq \infty, \quad t > \frac{1}{p_1}, \\ n^{-\frac{tp_1}{2}}(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_1})} & \text{if } 2 < p_1 < \infty, \quad t < \frac{1}{p_1}, \end{cases}$$

for all $n \geq 2$.

Remark 3.8. The most interesting case is given by $p_1 = 1$. It follows that we have

$$x_n(id : S_{1,1}^t B((0, 1)^d) \rightarrow L_1((0, 1)^d)) \asymp \begin{cases} n^{-t} & \text{if } t < \frac{1}{2}, \\ n^{-t}(\log n)^{(d-1)(t-\frac{1}{2})} & \text{if } t > \frac{1}{2}, \end{cases}$$

for all $n \geq 2$. We are not aware of any other result concerning s -numbers (Kolmogorov numbers, approximation numbers, ...) where the exact order of $s_n(id : S_{1,1}^t B((0, 1)^d) \rightarrow L_1((0, 1)^d))$ if n tends to infinity, has been found. A few more results concerning approximation and Kolmogorov numbers are known in case of the embeddings $id : S_{p_1, p_1}^t B((0, 1)^d) \rightarrow L_1((0, 1)^d)$, $p_1 > 1$, and $id : S_{1,1}^t B((0, 1)^d) \rightarrow L_{p_2}((0, 1)^d)$, $1 < p_2 < \infty$. E.g., in [44] Romanyuk has proved for $2 \leq p_1 < \infty$ and $t > 0$

$$a_n(id : S_{p_1, p_1}^t B((0, 1)^d) \rightarrow L_1((0, 1)^d)) \asymp n^{-t}(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_1})}$$

for all $n \geq 2$.

3.4 A version of Hölder-Zygmund spaces (related to tensor products) as target spaces

As a supplement we investigate the Weyl numbers of the embeddings $id : S_{p_1, p_1}^t B((0, 1)^d) \rightarrow \mathcal{Z}_{\text{mix}}^s((0, 1)^d)$, where the spaces $\mathcal{Z}_{\text{mix}}^s((0, 1)^d)$ are versions of Hölder-Zygmund spaces. Let $j \in \{1, \dots, d\}$. For $m \in \mathbb{N}$, $h_j \in \mathbb{R}$ and $x \in \mathbb{R}^d$ we put

$$\Delta_{h_j, j}^m f(x) := \sum_{\ell=0}^m (-1)^{m-\ell} \binom{m}{\ell} f(x_1, \dots, x_{j-1}, x_j + \ell h_j, x_{j+1}, \dots, x_d).$$

This is the m -th order difference in direction j . Mixed differences are defined as follows. Let e be a non-trivial subset of $\{1, \dots, d\}$. For $h \in \mathbb{R}^d$ we define

$$\Delta_{h, e}^m := \prod_{j \in e} \Delta_{h_j, j}^m.$$

Of course, here $\Delta_{h_j, j}^m \cdot \Delta_{h_\ell, \ell}^m$ has to be interpreted as $\Delta_{h_j, j}^m \circ \Delta_{h_\ell, \ell}^m$.

Definition 3.9. Let $s > 0$. Let $m \in \mathbb{N}$ s.t. $m - 1 \leq s < m$. Then $f \in \mathcal{Z}_{\text{mix}}^s((0, 1)^d)$ if

$$\|f| \mathcal{Z}_{\text{mix}}^s((0, 1)^d)\| := \|f| C((0, 1)^d)\| + \max_{e \subset \{1, \dots, d\}} \sup_{\|h\|_\infty \leq 1} \prod_{j \in e} |h_j|^{-s} \sup_{x \in \Omega_{m, e, h}} |\Delta_{h, e}^m f(x)| < \infty,$$

where

$$\Omega_{m,e,h} := \{x \in (0,1)^d : (x_1 + \varepsilon_1 \ell_1 h_1, \dots, x_d + \varepsilon_d \ell_d h_d) \in (0,1)^d \quad \forall \bar{\ell} \in \mathbb{N}_0^d, \|\bar{\ell}\|_\infty \leq m\},$$

and

$$\varepsilon_j := \begin{cases} 1 & \text{if } j \in e, \\ 0 & \text{if } j \notin e. \end{cases}$$

A few properties of these spaces are obvious:

- Let $d = 1$ and $m = 1$, i.e., $0 < s < 1$. Then $\mathcal{Z}_{\text{mix}}^s(0,1)$ is the classical space of Hölder-continuous functions of order s .
- Let $d = 1$, $s = 1$ and $m = 2$. Then $\mathcal{Z}_{\text{mix}}^1(0,1)$ is the classical Zygmund space.
- If $f(x) = f_1(x_1) \cdot \dots \cdot f_d(x_d)$ with $f_j \in \mathcal{Z}_{\text{mix}}(0,1)$, $j = 1, \dots, d$, then $f \in \mathcal{Z}_{\text{mix}}^s((0,1)^d)$ follows and

$$\|f|_{\mathcal{Z}_{\text{mix}}^s((0,1)^d)}\| \asymp \prod_{j=1}^d \|f_j|_{\mathcal{Z}_{\text{mix}}^s(0,1)}\|.$$

- Let $f \in \mathcal{Z}_{\text{mix}}^s((0,1)^d)$ and define $g(x') := f(x', 0)$, where $x = (x', x_d)$, $x' \in \mathbb{R}^{d-1}$. Then $g \in \mathcal{Z}_{\text{mix}}^s((0,1)^{d-1})$ follows.
- We define $\mathcal{Z}_{\text{mix}}^s(\mathbb{R}^d)$ by replacing $(0,1)^d$ by \mathbb{R}^d in the Def. 3.9. Let $E : \mathcal{Z}^s(0,1) \rightarrow \mathcal{Z}^s(\mathbb{R})$ be a linear and bounded extension operator such that E maps $C(0,1)$ into itself. Then $E \otimes \dots \otimes E$ (d -fold tensor product) is well-defined on $C((0,1)^d)$ and maps this space into itself. Observe that $\Delta_{h,e}^m f(x)$ can be written as the $|e|$ -fold iteration of a directional difference. As a consequence we obtain that $\mathcal{E}_d := E \otimes \dots \otimes E$ maps $\mathcal{Z}_{\text{mix}}^s((0,1)^d)$ into itself.

Less obvious is the following lemma, see [47, Rem. 2.3.4/3] or [60].

Lemma 3.10. *Let $s > 0$. Then*

$$\mathcal{Z}_{\text{mix}}^s((0,1)^d) = S_{\infty,\infty}^s B((0,1)^d)$$

holds in the sense of equivalent norms.

Essentially by the same methods as used for the proof of Thm. 3.1 one obtains the following.

Theorem 3.11. *Let $s > 0$ and $t > s + \frac{1}{p_1}$. Then it holds*

$$\begin{aligned} x_n(\text{id} : S_{p_1,p_1}^t B((0,1)^d) &\rightarrow \mathcal{Z}_{\text{mix}}^s((0,1)^d)) \\ &\asymp \begin{cases} \frac{(\log n)^{(d-1)(t-s-\frac{1}{p_1})}}{n^{t-s-\frac{1}{2}}} & \text{if } 0 < p_1 \leq 2, \\ \frac{(\log n)^{(d-1)(t-s-\frac{1}{p_1})}}{n^{t-s-\frac{1}{p_1}}} & \text{if } 2 \leq p_1 \leq \infty, \end{cases} \end{aligned}$$

for all $n \geq 2$.

Remark 3.12. (i) Recall that $S_{p_1, p_1}^t B((0, 1)^d)$ is compactly embedded into $\mathcal{Z}_{\text{mix}}^s((0, 1)^d)$ if and only if $t > s + 1/p_1$, see [61].

(ii) Observe, that Thm. 3.11 is not the limit of Thm. 3.4 for $s \downarrow 0$. There, in Thm. 3.4, is an additional factor $(\log n)^{(d-1)/2}$ as many times in this field.

(iii) If we replace $\mathcal{Z}_{\text{mix}}^s((0, 1)^d)$ by $S_{\infty, \infty}^s B((0, 1)^d)$ in Theorem 3.11, then the restriction $s > 0$ becomes superfluous.

Finally, we wish to mention that these methods also apply in case of approximation numbers. As a result we get the following.

Theorem 3.13. *Let $n \in \mathbb{N}, n \geq 2$ and $s > 0$. Then we have*

$$a_n(\text{id} : S_{p_1, p_1}^t B((0, 1)^d) \rightarrow \mathcal{Z}_{\text{mix}}^s((0, 1)^d)) \asymp \begin{cases} \frac{(\log n)^{(d-1)(t-s-\frac{1}{p_1})}}{n^{t-s-\frac{1}{p_1}}} & \text{if } 2 \leq p_1 \leq \infty, \quad t-s > \frac{1}{p_1}, \\ \frac{(\log n)^{(d-1)(t-s-\frac{1}{p_1})}}{n^{t-s-\frac{1}{2}}} & \text{if } 1 \leq p_1 < 2, \quad t-s > 1, \\ \frac{(\log n)^{(d-1)(t-s-\frac{1}{p_1})}}{n^{\frac{p_1'}{2}(t-s-\frac{1}{p_1})}} & \text{if } 1 < p_1 < 2, \quad 1 > t-s > \frac{1}{p_1}, \end{cases}$$

for all $n \geq 2$. Here p_1' is the conjugate of p_1 .

Remark 3.14. As in Remark 3.12, if we replace $\mathcal{Z}_{\text{mix}}^s((0, 1)^d)$ by $S_{\infty, \infty}^s B((0, 1)^d)$ in Theorem 3.13, then the restriction $s > 0$ becomes superfluous.

3.5 A comparison with entropy numbers

There are good reasons to compare Weyl numbers with entropy numbers. Both, entropy and Weyl numbers, are tools to control the behaviour of eigenvalues of linear operators.

Let us recall the definition of entropy numbers.

Definition 3.15. *Let $T : X \rightarrow Y$ be a bounded linear operator between complex quasi-Banach spaces, and let $n \in \mathbb{N}$. Then the n -th (dyadic) entropy number of T is defined as*

$$e_n(T : X \rightarrow Y) := \inf\{\varepsilon > 0 : T(B_X) \text{ can be covered by } 2^{n-1} \text{ balls in } Y \text{ of radius } \varepsilon\},$$

where $B_X := \{x \in X : \|x\|_X \leq 1\}$ denotes the closed unit ball of X .

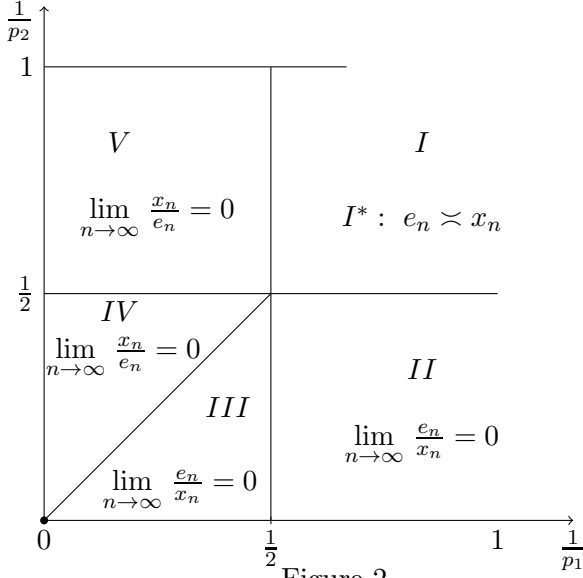
In particular, $T : X \rightarrow Y$ is compact if and only if $\lim_{n \rightarrow \infty} e_n(T) = 0$. For details and basic properties like multiplicativity, additivity, behaviour under interpolation etc. we refer to the monographs [12, 18, 27, 32]. Most important for us is the Carl-Triebel inequality which states

$$|\lambda_n(T)| \leq \sqrt{2} e_n(T),$$

cf. Carl, Triebel [13] (see also the monographs [12] and [18]).

Entropy numbers of embeddings $id : S_{p_1, p_1}^t B((0, 1)^d) \hookrightarrow L_{p_2}((0, 1)^d)$ have been investigated in Vybiral [61]. The picture is less complete than in case of Weyl numbers. Only for sufficiently large smoothness the behaviour is exactly known. For $0 < p_1 \leq \infty$ and $1 < p_2 < \infty$ we have

$$e_n(id) \asymp n^{-t(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_1})}} \quad \text{if} \quad t > \max\left(0, \frac{1}{p_1} - \frac{1}{2}, \frac{1}{p_1} - \frac{1}{p_2}\right), \quad n \geq 2.$$



We use Figure 2 to explain the different behaviour of entropy and Weyl numbers. Weyl numbers are essentially smaller than entropy numbers in regions IV and V, entropy numbers are essentially smaller than Weyl numbers in regions II and III, and they show a similar behaviour in region I*.

Remark 3.16. (i) Further estimates of the decay of entropy numbers related to embeddings $id : S_{p_1, q_1}^t B((0, 1)^d) \hookrightarrow L_{p_2}((0, 1)^d)$ ($id : S_{p_1}^t W((0, 1)^d) \hookrightarrow L_{p_2}((0, 1)^d)$) can be found in Belinsky [6], Dinh Dũng [16], and Temlyakov [51].

(ii) There are many contributions dealing with the behaviour of $d_n(id : S_{p_1, q_1}^t B((0, 1)^d) \hookrightarrow L_{p_2}((0, 1)^d))$ (Kolmogorov numbers) and $a_n(id : S_{p_1, q_1}^t B((0, 1)^d) \hookrightarrow L_{p_2}((0, 1)^d))$. However, the picture is much less complete than in case of Weyl numbers. We refer to Bazarkhanov [5] for the most recent publication in this direction. The topic itself has been investigated at various places over the last 30 years, see, e.g., Temlyakov [51, 52], Galeev [19, 20, 21] and Romanyuk [36, 37, 38, 39, 40, 41, 42, 43, 44].

4 Weyl numbers - basic properties

Weyl numbers are special s -numbers. For later use we recall this general notion following Pietsch [35, 2.2.1] (note that this differs slightly from earlier definitions in the literature).

Let X, Y, X_0, Y_0 be quasi-Banach spaces. As usual, $\mathcal{L}(X, Y)$ denotes the space of all continuous linear operators from X to Y . Finally, let Y be p -Banach space for some

$p \in (0, 1]$, i.e.,

$$\|x + y\|_Y^p \leq \|x\|_Y^p + \|y\|_Y^p \quad \text{for all } x, y \in Y. \quad (4.1)$$

An s -function is a map s assigning to every operator $T \in \mathcal{L}(X, Y)$ a scalar sequence $(s_n(T))$ such that the following conditions are satisfied:

- (a) $\|T\| = s_1(T) \geq s_2(T) \geq \dots \geq 0$ for all $T \in \mathcal{L}(X, Y)$;
- (b) $s_{n+m-1}^p(S + T) \leq s_n^p(S) + s_m^p(T)$ for $S, T \in \mathcal{L}(X, Y)$ and $m, n = 1, 2, \dots$;
- (c) $s_n(BTA) \leq \|B\| \cdot s_n(T) \cdot \|A\|$ for $A \in \mathcal{L}(X_0, X)$, $T \in \mathcal{L}(X, Y)$, $B \in \mathcal{L}(Y, Y_0)$;
- (d) $s_n(T) = 0$ if $\text{rank}(T) < n$ for all $n \in \mathbb{N}$;
- (e) $s_n(\text{id} : \ell_2^n \rightarrow \ell_2^n) = 1$ for all $n \in \mathbb{N}$.

We will refer to (a) as monotonicity, to (b) as additivity, to (c) as ideal property, to (d) as the rank property and to (e) as normalization (norm-determining property) of the s -numbers.

Sometimes a further property is of some use. Let Z be a quasi-Banach space. An s -function is called multiplicative if

- (f) $s_{n+m-1}(ST) \leq s_n(S) s_m(T)$ for $T \in \mathcal{L}(X, Y)$, $S \in \mathcal{L}(Y, Z)$ and $m, n = 1, 2, \dots$.

Examples

The following numbers are s -numbers:

- (i) Kolmogorov numbers are multiplicative s -numbers, see, e.g., [32, Thm. 11.9.2].
- (ii) Approximation numbers are multiplicative s -numbers, see, e.g., [35, 2.3.3].
- (iii) The n -th Gelfand number of the linear operator $T \in \mathcal{L}(X, Y)$ is defined to be

$$c_n(T) := \inf \left\{ \|T J_M^X\| : \text{codim}(M) < n \right\},$$

where $J_M^X : M \rightarrow X$ refers to the canonical injection of M into X . Gelfand numbers are multiplicative s -numbers, see, e.g., [35, Prop. 2.4.8].

- (iv) Weyl numbers are multiplicative s -numbers, see [35, 2.4.14, 2.4.17].

Entropy numbers do not belong to the class of s -numbers since they do not satisfy (d).

Remark 4.1. There is an alternative way to calculate the n -th Weyl number. Indeed, for $T \in \mathcal{L}(X, Y)$ it holds

$$x_n(T) := \sup \left\{ c_n(TA) : A \in \mathcal{L}(\ell_2, X), \|A\| \leq 1 \right\},$$

see Pietsch [33].

Interpolation properties of Weyl numbers

For later use we add the following assertion concerning interpolation properties of Weyl numbers.

Theorem 4.2. *Let $0 < \theta < 1$. Let X, Y, Y_0, Y_1 be a quasi-Banach spaces. Further we assume $Y_0 \cap Y_1 \hookrightarrow Y$ and the existence of a positive constant C such that*

$$\|y|Y\| \leq C \|y|Y_0\|^{1-\theta} \|y|Y_1\|^\theta \quad \text{for all } y \in Y_0 \cap Y_1. \quad (4.2)$$

Then, if

$$T \in \mathcal{L}(X, Y_0) \cap \mathcal{L}(X, Y_1) \cap \mathcal{L}(X, Y)$$

we obtain

$$x_{n+m-1}(T : X \rightarrow Y) \leq C x_n^{1-\theta}(T : X \rightarrow Y_0) x_m^\theta(T : X \rightarrow Y_1)$$

for all $n, m \in \mathbb{N}$. Here C is the same constant as in (4.2).

Remark 4.3. Interpolation properties of Kolmogorov and Gelfand numbers have been studied by Triebel [55]. Theorem 4.2 and Theorem 7.9 below show that Gelfand and Weyl numbers share the same interpolation properties.

5 Tensor product Besov spaces and spaces of dominating mixed smoothness

As mentioned before tensor product Besov spaces can be interpreted as special cases of the scale of Besov spaces of dominating mixed smoothness. For us it will be convenient to introduce these classes of dominating mixed smoothness by means of wavelets. In the Appendix B below we recall the probably better known Fourier-analytic definition. In addition we shall introduce Lizorkin-Triebel spaces of dominating mixed smoothness. They will be used in our proofs of the main results for Besov spaces.

Let $\bar{\nu} = (\nu_1, \dots, \nu_d) \in \mathbb{N}_0^d$ and $\bar{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$. Then we put $2^{-\bar{\nu}}\bar{m} = (2^{-\nu_1}m_1, \dots, 2^{-\nu_d}m_d)$ and

$$Q_{\bar{\nu}, \bar{m}} := \left\{ x \in \mathbb{R}^d : 2^{-\nu_\ell} m_\ell < x_\ell < 2^{-\nu_\ell} (m_\ell + 1), \ell = 1, \dots, d \right\}.$$

By $\chi_{\bar{\nu}, \bar{m}}(x)$ we denote the characteristic function of $Q_{\bar{\nu}, \bar{m}}$. First we have to introduce some sequence spaces.

Definition 5.1. *If $0 < p, q \leq \infty$, $t \in \mathbb{R}$ and $\lambda := \{\lambda_{\bar{\nu}, \bar{m}} \in \mathbb{C} : \bar{\nu} \in \mathbb{N}_0^d, \bar{m} \in \mathbb{Z}^d\}$, then we define*

$$s_{p,q}^t b := \left\{ \lambda : \|\lambda|s_{p,q}^t b\| = \left(\sum_{\bar{\nu} \in \mathbb{N}_0^d} 2^{|\bar{\nu}|_1(t-\frac{1}{p})q} \left(\sum_{\bar{m} \in \mathbb{Z}^d} |\lambda_{\bar{\nu}, \bar{m}}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \right\}$$

and, if $p < \infty$,

$$s_{p,q}^t f = \left\{ \lambda : \|\lambda\|_{s_{p,q}^t f} = \left\| \left(\sum_{\bar{\nu} \in \mathbb{N}_0^d} \sum_{\bar{m} \in \mathbb{Z}^d} |2^{|\bar{\nu}|_1 t} \lambda_{\bar{\nu}, \bar{m}} \chi_{\bar{\nu}, \bar{m}}(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L_p(\mathbb{R}^d)} < \infty \right\}$$

with the usual modification for p or/and q equal to ∞ .

Remark 5.2. Let $\sigma \in \mathbb{R}$. For later use we mention that the mapping

$$J_\sigma : (\lambda_{\bar{\nu}, \bar{m}})_{\bar{\nu}, \bar{m}} \mapsto (2^{\sigma|\bar{\nu}|_1} \lambda_{\bar{\nu}, \bar{m}})_{\bar{\nu}, \bar{m}} \quad (5.1)$$

yields an isomorphism of $s_{p,q}^t a$ onto $s_{p,q}^{t-\sigma} a$, $a \in \{b, f\}$.

Now we recall wavelet bases of Besov and Lizorkin-Triebel spaces of dominating mixed smoothness. Let $N \in \mathbb{N}$. Then there exists $\psi_0, \psi_1 \in C^N(\mathbb{R})$, compactly supported,

$$\int_{-\infty}^{\infty} t^m \psi_1(t) dt = 0, \quad m = 0, 1, \dots, N,$$

such that $\{2^{j/2} \psi_{j,m} : j \in \mathbb{N}_0, m \in \mathbb{Z}\}$, where

$$\psi_{j,m}(t) := \begin{cases} \psi_0(t-m) & \text{if } j = 0, m \in \mathbb{Z}, \\ \sqrt{1/2} \psi_1(2^{j-1}t - m) & \text{if } j \in \mathbb{N}, m \in \mathbb{Z}, \end{cases}$$

is an orthonormal basis in $L_2(\mathbb{R})$, see [63]. We put

$$\Psi_{\bar{\nu}, \bar{m}}(x) := \prod_{\ell=1}^d \psi_{\nu_\ell, m_\ell}(x_\ell).$$

Then

$$\Psi_{\bar{\nu}, \bar{m}}, \quad \bar{\nu} \in \mathbb{N}_0^d, \bar{m} \in \mathbb{Z}^d,$$

is a tensor product wavelet basis of $L_2(\mathbb{R}^d)$. Vybiral [61] has proved the following.

Lemma 5.3. *Let $0 < p, q \leq \infty$ and $t \in \mathbb{R}$.*

(i) *There exists $N = N(t, p) \in \mathbb{N}$ s.t. the mapping*

$$\mathcal{W} : f \mapsto (2^{|\bar{\nu}|_1} \langle f, \Psi_{\bar{\nu}, \bar{m}} \rangle)_{\bar{\nu} \in \mathbb{N}_0^d, \bar{m} \in \mathbb{Z}^d}$$

is an isomorphism of $S_{p,q}^t B(\mathbb{R}^d)$ onto $s_{p,q}^t b$.

(ii) *Let $p < \infty$. Then there exists $N = N(t, p, q) \in \mathbb{N}$ s.t. the mapping \mathcal{W} is an isomorphism of $S_{p,q}^t F(\mathbb{R}^d)$ onto $s_{p,q}^t f$.*

Spaces on Ω

We put $\Omega := (0, 1)^d$. For us it will be convenient to define spaces on Ω by restrictions. We shall need the set $D'(\Omega)$, consisting of all complex-valued distributions on Ω .

Definition 5.4. (i) Let $0 < p, q \leq \infty$ and $t \in \mathbb{R}$. Then $S_{p,q}^t B((0,1)^d)$ is the space of all $f \in D'(\Omega)$ such that there exists a distribution $g \in S_{p,q}^t B(\mathbb{R}^d)$ satisfying $f = g|_\Omega$. It is endowed with the quotient norm

$$\|f|_{S_{p,q}^t B((0,1)^d)}\| = \inf \left\{ \|g|_{S_{p,q}^t B(\mathbb{R}^d)}\| : g|_\Omega = f \right\}.$$

(ii) Let $0 < p < \infty$, $0 < q \leq \infty$ and $t \in \mathbb{R}$. Then $S_{p,q}^t F((0,1)^d)$ is the space of all $f \in D'(\Omega)$ such that there exists a distribution $g \in S_{p,q}^t F(\mathbb{R}^d)$ satisfying $f = g|_\Omega$. It is endowed with the quotient norm

$$\|f|_{S_{p,q}^t F((0,1)^d)}\| = \inf \left\{ \|g|_{S_{p,q}^t F(\mathbb{R}^d)}\| : g|_\Omega = f \right\}.$$

Several times we shall work with the following consequence of this definition in combination with Lemma 5.3. Let t, p and q be fixed. Let the wavelet basis $\Psi_{\bar{\nu}, \bar{m}}$ be admissible in the sense of Lemma 5.3. We put

$$A_{\bar{\nu}}^\Omega := \left\{ \bar{m} \in \mathbb{Z}^d : \text{supp } \Psi_{\bar{\nu}, \bar{m}} \cap \Omega \neq \emptyset \right\}, \quad \bar{\nu} \in \mathbb{N}_0^d. \quad (5.2)$$

For given $f \in S_{p,q}^t A(\Omega)$, $A \in \{B, F\}$, let $\mathcal{E}f$ be an element of $S_{p,q}^t A(\mathbb{R}^d)$ s.t.

$$\|\mathcal{E}f|_{S_{p,q}^t A(\mathbb{R}^d)}\| \leq 2 \|f|_{S_{p,q}^t A(\Omega)}\| \quad \text{and} \quad (\mathcal{E}f)|_\Omega = f.$$

We define

$$g := \sum_{\bar{\nu} \in \mathbb{N}_0^d} \sum_{\bar{m} \in A_{\bar{\nu}}^\Omega} 2^{|\bar{\nu}|_1} \langle \mathcal{E}f, \Psi_{\bar{\nu}, \bar{m}} \rangle \Psi_{\bar{\nu}, \bar{m}}.$$

Then it follows that $g \in S_{p,q}^t A(\mathbb{R}^d)$, $g|_\Omega = f$,

$$\text{supp } g \subset \{x \in \mathbb{R}^d : \max_{j=1, \dots, d} |x_j| \leq c_1\} \quad \text{and} \quad \|g|_{S_{p,q}^t A(\mathbb{R}^d)}\| \leq c_2 \|f|_{S_{p,q}^t A(\Omega)}\|.$$

Here c_1, c_2 are independent of f .

Tensor products of Besov spaces

Tensor products of Besov spaces have been investigated in [25], [48] and [49]. We recall some results from [48] and [49]. For the basic notions of tensor products used here we refer to [28] and [15]. By α_p we denote the p -nuclear norm and by γ_p the projective tensor p -norm.

Theorem 5.5 (Tensor products of Besov spaces on the interval).

Let $d \geq 1$ and let $t \in \mathbb{R}$.

(i) Let $1 < p < \infty$. Then the following formula

$$\begin{aligned} B_{p,p}^t(0,1) \otimes_{\alpha_p} S_{p,p}^t B((0,1)^d) &= S_{p,p}^t B((0,1)^d) \otimes_{\alpha_p} B_{p,p}^t(0,1) \\ &= S_{p,p}^t B((0,1)^{d+1}) \end{aligned}$$

holds true in the sense of equivalent norms.

(ii) Let $0 < p \leq 1$. Then the following formula

$$\begin{aligned} B_{p,p}^t(0,1) \otimes_{\gamma_p} S_{p,p}^t B((0,1)^d) &= S_{p,p}^t B((0,1)^d) \otimes_{\gamma_p} B_{p,p}^t(0,1) \\ &= S_{p,p}^t B((0,1)^{d+1}) \end{aligned}$$

holds true in the sense of equivalent quasi-norms.

Remark 5.6. For easier notation we put $\gamma_p := \alpha_p$ if $1 < p < \infty$. One can iterate the process of taking tensor products. Defining for $m > 2$

$$X_1 \otimes_{\gamma_p} X_2 \otimes_{\gamma_p} \dots \otimes_{\gamma_p} X_m := X_1 \otimes_{\gamma_p} \left(\dots X_{m-2} \otimes_{\gamma_p} (X_{m-1} \otimes_{\gamma_p} X_m) \right)$$

we obtain an interpretation of $S_{p,p}^t B((0,1)^d)$, $0 < p < \infty$, as an iterated tensor product of univariate Besov spaces, namely

$$S_{p,p}^t B((0,1)^d) = B_{p,p}^t(0,1) \otimes_{\gamma_p} \dots \otimes_{\gamma_p} B_{p,p}^t(0,1), \quad 0 < p < \infty.$$

The iterated tensor products, considered in this paper, do not depend on the order of the tuples which are formed during the process of calculating $X_1 \otimes_{\gamma_p} X_2 \otimes_{\gamma_p} \dots \otimes_{\gamma_p} X_m$, i.e.,

$$(X_1 \otimes_{\gamma_p} X_2) \otimes_{\gamma_p} X_3 = X_1 \otimes_{\gamma_p} (X_2 \otimes_{\gamma_p} X_3).$$

Consequently, if $p < \infty$, we may deal with $S_{p,p}^t B((0,1)^d)$ instead of $B_{p,p}^t(0,1) \otimes_{\gamma_p} \dots \otimes_{\gamma_p} B_{p,p}^t(0,1)$.

6 Weyl numbers of embeddings of sequence spaces

In this section we will estimate the behavior of Weyl numbers of the identity mapping

$$id^* : s_{p_0,p_0}^{t,\Omega} b \rightarrow s_{p,2}^{0,\Omega} f.$$

Here we assume that p_0 varies in $(0, \infty]$ and p in $(0, \infty)$.

6.1 Preparations

For technical reasons we need a few more sequence spaces. Recall, $A_{\bar{\nu}}^{\Omega}$ has been defined in (5.2).

Definition 6.1. If $0 < p \leq \infty$, $0 < q \leq \infty$, $t \in \mathbb{R}$ and

$$\lambda = \{\lambda_{\bar{\nu}, \bar{m}} \in \mathbb{C} : \bar{\nu} \in \mathbb{N}_0^d, \bar{m} \in A_{\bar{\nu}}^{\Omega}\},$$

then we define

$$s_{p,q}^{t,\Omega} b := \left\{ \lambda : \|\lambda|s_{p,q}^{t,\Omega} b\| = \left(\sum_{\bar{\nu} \in \mathbb{N}_0^d} 2^{|\bar{\nu}|_1(t-\frac{1}{p})q} \left(\sum_{\bar{m} \in A_{\bar{\nu}}^{\Omega}} |\lambda_{\bar{\nu}, \bar{m}}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \right\}$$

and, if $p < \infty$,

$$s_{p,q}^{t,\Omega} f := \left\{ \lambda : \|\lambda|s_{p,q}^{t,\Omega} f\| = \left\| \left(\sum_{\bar{\nu} \in \mathbb{N}_0^d} \sum_{\bar{m} \in A_{\bar{\nu}}^\Omega} |2^{|\bar{\nu}|_1 t} \lambda_{\bar{\nu}, \bar{m}} \chi_{\bar{\nu}, \bar{m}}(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L_p(\mathbb{R}^d)} \right\}.$$

In addition we need the following sequence of subspaces.

Definition 6.2. If $0 < p \leq \infty$, $0 < q \leq \infty$, $t \in \mathbb{R}$, $\mu \in \mathbb{N}_0$ and

$$\lambda = \{\lambda_{\bar{\nu}, \bar{m}} \in \mathbb{C} : \bar{\nu} \in \mathbb{N}_0^d, |\bar{\nu}|_1 = \mu, \bar{m} \in A_{\bar{\nu}}^\Omega\},$$

then we define

$$(s_{p,q}^{t,\Omega} b)_\mu = \left\{ \lambda : \|\lambda|(s_{p,q}^{t,\Omega} b)_\mu\| = \left(\sum_{|\bar{\nu}|_1 = \mu} 2^{|\bar{\nu}|_1(t - \frac{1}{p})q} \left(\sum_{\bar{m} \in A_{\bar{\nu}}^\Omega} |\lambda_{\bar{\nu}, \bar{m}}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \right\}$$

and, if $p < \infty$,

$$(s_{p,q}^{t,\Omega} f)_\mu = \left\{ \lambda : \|\lambda|(s_{p,q}^{t,\Omega} f)_\mu\| = \left\| \left(\sum_{|\bar{\nu}|_1 = \mu} \sum_{\bar{m} \in A_{\bar{\nu}}^\Omega} |2^{|\bar{\nu}|_1 t} \lambda_{\bar{\nu}, \bar{m}} \chi_{\bar{\nu}, \bar{m}}(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L_p(\mathbb{R}^d)} \right\}.$$

To avoid repetitions we shall use $s_{p,q}^t a$, $s_{p,q}^{t,\Omega} a$, $(s_{p,q}^{t,\Omega} a)_\mu$ with $a \in \{b, f\}$ in case that an assertion holds for both scales simultaneously. Here in this paper we do not deal with the spaces $s_{\infty,q}^{t,\Omega} f$ and $(s_{\infty,q}^{t,\Omega} f)_\mu$. But we will use the convention that, whenever $s_{\infty,q}^{t,\Omega} a$ or $(s_{\infty,q}^{t,\Omega} a)_\mu$ occur, this has to be interpreted as $s_{\infty,q}^{t,\Omega} b$ and $(s_{\infty,q}^{t,\Omega} b)_\mu$. The two following elementary lemmas are taken from [61, Lemma 3.10] and [24, Lemma 6.4.2].

Lemma 6.3. (i) We have

$$|A_{\bar{\nu}}^\Omega| \asymp 2^{|\bar{\nu}|_1}, \quad D_\mu := \sum_{|\bar{\nu}|_1 = \mu} |A_{\bar{\nu}}^\Omega| \asymp \mu^{d-1} 2^\mu$$

with equivalence constants independent of $\bar{\nu} \in \mathbb{N}_0^d$ and $\mu \in \mathbb{N}_0$.

(ii) Let $0 < p < \infty$ and $t \in \mathbb{R}$. Then

$$s_{p,p}^{t,\Omega} f = s_{p,p}^{t,\Omega} b$$

and

$$(s_{p,p}^{t,\Omega} f)_\mu = (s_{p,p}^{t,\Omega} b)_\mu = 2^{\mu(t - \frac{1}{p})} \ell_p^{D_\mu}, \quad \mu \in \mathbb{N}_0,$$

with the obvious interpretation for the quasi-norms.

Lemma 6.4. (i) Let $0 < p_0, p \leq \infty$ and $0 < q \leq \infty$. Then

$$\|id_\mu^* : (s_{p_0,q}^{t,\Omega} a)_\mu \rightarrow (s_{p,q}^{t,\Omega} a)_\mu\| \asymp 2^{\mu(\frac{1}{p_0} - \frac{1}{p})_+}$$

with equivalence constants independent of $\mu \in \mathbb{N}_0$.

(ii) Let $0 < q_0, q \leq \infty$ and $0 < p \leq \infty$. Then

$$\|id_\mu^* : (s_{p,q_0}^{t,\Omega} a)_\mu \rightarrow (s_{p,q}^{t,\Omega} a)_\mu\| \asymp \mu^{(d-1)(\frac{1}{q}-\frac{1}{q_0})_+}$$

with equivalence constants independent of $\mu \in \mathbb{N}_0$.

Corollary 6.5. Let $0 < p_0, p, q_0, q \leq \infty$ and $t \in \mathbb{R}$. Then

$$\|id_\mu^* : (s_{p_0,q_0}^{t,\Omega} a)_\mu \rightarrow (s_{p,q}^{0,\Omega} a)_\mu\| \lesssim 2^{\mu(-t+(\frac{1}{p_0}-\frac{1}{p})_+)} \mu^{(d-1)(\frac{1}{q}-\frac{1}{q_0})_+},$$

with a constant behind \lesssim independent of μ .

Proof. This is an immediate consequence of Lemma 6.4. ■

Sometimes the previous estimate can be improved.

Lemma 6.6. Let $0 < p_0 < p < \infty$, $0 < q_0, q \leq \infty$ and $t \in \mathbb{R}$. Then

$$\|id_\mu^* : (s_{p_0,q_0}^{t,\Omega} f)_\mu \rightarrow (s_{p,q}^{0,\Omega} f)_\mu\| \lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}.$$

Proof. This assertion is contained in [24]. Since this phd is not published we give a proof. Let λ be a sequence such that $\lambda_{\bar{\nu}, \bar{m}} = 0$ if $|\bar{\nu}|_1 \neq \mu$. Since $p_0 < p$ the Sobolev-type embedding yields

$$s_{p_0,q_0}^{t,\Omega} f \hookrightarrow s_{p,q}^{t-\frac{1}{p_0}+\frac{1}{p},\Omega} f,$$

see [47, Thm. 2.4.1] ($d = 2$) or [26], we have

$$\begin{aligned} \|\lambda|(s_{p,q}^{0,\Omega} f)_\mu\| &= \|\lambda|s_{p,q}^{0,\Omega} f\| = 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} \|\lambda|s_{p,q}^{t-\frac{1}{p_0}+\frac{1}{p},\Omega} f\| \\ &\lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} \|\lambda|s_{p_0,q_0}^{t,\Omega} f\| = 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} \|\lambda|(s_{p_0,q_0}^{t,\Omega} f)_\mu\|. \end{aligned}$$

This proves the claim. ■

6.2 Weyl numbers of embeddings of sequence spaces related to spaces of dominating mixed smoothness - preparations

For $\mu \in \mathbb{N}_0$ we define

$$id_\mu^* : s_{p_0,p_0}^{t,\Omega} b \rightarrow s_{p,2}^{0,\Omega} f,$$

where

$$(id_\mu^* \lambda)_{\bar{\nu}, \bar{m}} := \begin{cases} \lambda_{\bar{\nu}, \bar{m}} & \text{if } |\bar{\nu}|_1 = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

The main idea of our proof is the following splitting of $id^* : s_{p_0,p_0}^{t,\Omega} b \rightarrow s_{p,2}^{0,\Omega} f$ into a sum of identities between building blocks

$$id^* = \sum_{\mu=0}^{\infty} id_\mu^* = \sum_{\mu=0}^J id_\mu^* + \sum_{\mu=J+1}^L id_\mu^* + \sum_{\mu=L+1}^{\infty} id_\mu^*, \quad (6.1)$$

where J and L are at our disposal. These numbers J and L will be chosen in dependence on the parameters. Let us mention that a similar splitting has been used by Vybiral [61] for the estimates of related entropy numbers.

The additivity and the monotonicity of the Weyl numbers and the quasi-triangle inequality (4.1) yield

$$x_n^\rho(id^*) \leq \sum_{\mu=0}^J x_{n_\mu}^\rho(id_\mu^*) + \sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) + \sum_{\mu=L+1}^{\infty} \|id_\mu^*\|^\rho, \quad \rho := \min(1, p), \quad (6.2)$$

where $n - 1 = \sum_{\mu=0}^L (n_\mu - 1)$. Of course, $\|id_\mu^*\| = \|id_\mu^* : (s_{p_0, p_0}^{t, \Omega} f)_\mu \rightarrow (s_{p, 2}^{0, \Omega} f)_\mu\|$. For brevity we put

$$\alpha = t - \left(\frac{1}{p_0} - \frac{1}{p} \right)_+.$$

Then by Corollary 6.5, we have

$$\|id_\mu^*\| \lesssim 2^{-\mu\alpha} \mu^{(d-1)(\frac{1}{2} - \frac{1}{p_0})_+},$$

which results in the estimate

$$\sum_{\mu=L+1}^{\infty} \|id_\mu^*\|^\rho \lesssim 2^{-L\alpha\rho} L^{(d-1)\rho(\frac{1}{2} - \frac{1}{p_0})_+}. \quad (6.3)$$

Now we choose n_μ

$$n_\mu := D_\mu + 1, \quad \mu = 0, 1, \dots, J. \quad (6.4)$$

Then we get

$$\sum_{\mu=0}^J n_\mu \asymp \sum_{\mu=0}^J \mu^{(d-1)} 2^\mu \asymp J^{d-1} 2^J \quad (6.5)$$

and $x_{n_\mu}(id_\mu^*) = 0$, see the rank property of the s -numbers, which implies

$$\sum_{\mu=0}^J x_{n_\mu}^\rho(id_\mu^*) = 0. \quad (6.6)$$

Summarizing (6.2)-(6.6) we have found

$$x_n^\rho(id^*) \lesssim \sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) + 2^{-L\alpha\rho} L^{(d-1)\rho(\frac{1}{2} - \frac{1}{p_0})_+}. \quad (6.7)$$

Now we turn to the problem to reduce the estimates for the Weyl numbers $x_{n_\mu}(id_\mu^*)$ to estimates for $x_n(id_{p_0, p}^m)$.

Proposition 6.7. *Let $0 < p_0 \leq \infty$ and $t \in \mathbb{R}$. Then we have the following assertions.*

(i) *If $0 < p \leq 2$, then*

$$\mu^{(d-1)(-\frac{1}{p} + \frac{1}{2})} 2^{\mu(-t + \frac{1}{p_0} - \frac{1}{p})} x_n(id_{p_0, p}^{D_\mu}) \lesssim x_n(id_\mu^*) \lesssim 2^{\mu(-t + \frac{1}{p_0} - \frac{1}{2})} x_n(id_{p_0, 2}^{D_\mu}). \quad (6.8)$$

(ii) If $2 \leq p < \infty$, then

$$2^{\mu(-t+\frac{1}{p_0}-\frac{1}{2})} x_n(id_{p_0,2}^{D_\mu}) \lesssim x_n(id_\mu^*) \lesssim \mu^{(d-1)(\frac{1}{2}-\frac{1}{p})} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id_{p_0,p}^{D_\mu}). \quad (6.9)$$

Proof. *Step 1.* Estimate from above. We define $\delta := \max(p, 2)$ and consider the following diagram:

$$\begin{array}{ccc} (s_{p_0,p_0}^{t,\Omega} b)_\mu & \xrightarrow{id_\mu^*} & (s_{p,2}^{0,\Omega} f)_\mu \\ & \searrow id^2 \quad \nearrow id^1 & \\ & (s_{\delta,\delta}^{0,\Omega} f)_\mu & \end{array}$$

Using property (c) of the s -numbers we conclude

$$x_n(id_\mu^*) \leq \|id^1\| x_n(id^2).$$

By Corollary 6.5, we have

$$\|id^1\| \lesssim \mu^{(d-1)(\frac{1}{2}-\frac{1}{\delta})}.$$

From Lemma 6.3 (ii), we derive

$$x_n(id^2) \lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{\delta})} x_n(id_{p_0,\delta}^{D_\mu}),$$

taking into account property (c) of the s -numbers and the commutative diagram

$$\begin{array}{ccc} 2^{\mu(t-\frac{1}{p_0})} (\ell_{p_0}^{D_\mu}) & \xrightarrow{id^3} & \ell_{p_0}^{D_\mu} \\ id^2 \downarrow & & \downarrow id_{p_0,\delta}^{D_\mu} \\ 2^{-\frac{\mu}{\delta}} (\ell_\delta^{D_\mu}) & \xleftarrow{id^4} & \ell_\delta^{D_\mu}, \end{array}$$

$$\text{i.e., } id^2 = id^4 \circ id_{p_0,\delta}^{D_\mu} \circ id^3,$$

$$\|id^3\| = 2^{-\mu(t-\frac{1}{p_0})} \quad \text{and} \quad \|id^4\| = 2^{\frac{\mu}{\delta}}.$$

Altogether this implies

$$x_n(id_\mu^*) \lesssim \mu^{(d-1)(\frac{1}{2}-\frac{1}{\delta})} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{\delta})} x_n(id_{p_0,\delta}^{D_\mu}).$$

Step 2. Now we turn to the estimate from below. We define $\gamma := \min(p, 2)$ and use the following commutative diagram

$$\begin{array}{ccc} (s_{p_0,p_0}^{t,\Omega} b)_\mu & \xrightarrow{id^2} & (s_{\gamma,\gamma}^{0,\Omega} f)_\mu \\ & \searrow id_\mu^* \quad \nearrow id^1 & \\ & (s_{p,2}^{0,\Omega} f)_\mu & \end{array}$$

This time we have $x_n(id_2) \leq \|id^1\| x_n(id_\mu^*)$ and by Corollary 6.5, we get

$$\|id^1\| \lesssim \mu^{(d-1)(\frac{1}{\gamma}-\frac{1}{2})}.$$

Similarly as in Step 1 Lemma 6.3 (ii) yields

$$x_n(id^2) \gtrsim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{\gamma})} x_n(id_{p_0,\gamma}^{D_\mu}).$$

Inserting this in our previous estimate we find

$$x_n(id_\mu^*) \gtrsim \mu^{(d-1)(\frac{1}{2}-\frac{1}{\gamma})} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{\gamma})} x_n(id_{p_0,\gamma}^{D_\mu}).$$

The proof is complete. ■

Proposition 6.8. *Let $0 < p_0 \leq \infty$ and $t \in \mathbb{R}$. Then we have the following assertions.*

(i) *If $0 < p \leq 2$, then*

$$x_n(id_\mu^*) \lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id_{p_0,p}^{D_\mu}). \quad (6.10)$$

(ii) *If $2 \leq p < \infty$, then*

$$2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id_{p_0,p}^{D_\mu}) \lesssim x_n(id_\mu^*). \quad (6.11)$$

Proof. *Step 1.* Proof of (i). We consider the following diagram

$$\begin{array}{ccc} (s_{p_0,p_0}^{t,\Omega} b)_\mu & \xrightarrow{id_\mu^*} & (s_{p,2}^{0,\Omega} f)_\mu \\ & \searrow id^2 & \nearrow id^1 \\ & (s_{p,p}^{0,\Omega} f)_\mu & \end{array}$$

This implies $x_n(id_\mu^*) \leq \|id^1\| x_n(id^2)$. Corollary 6.5 yields $\|id^1\| \lesssim 1$ and from Lemma 6.3 we derive

$$x_n(id^2) \lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id_{p_0,p}^{D_\mu}).$$

Altogether we have found

$$x_n(id_\mu^*) \lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id_{p_0,p}^{D_\mu}).$$

Step 2. Proof of (ii). We use the following diagram

$$\begin{array}{ccc} (s_{p_0,p_0}^{t,\Omega} b)_\mu & \xrightarrow{id^2} & (s_{p,p}^{0,\Omega} f)_\mu \\ & \searrow id_\mu^* & \nearrow id^1 \\ & (s_{p,2}^{0,\Omega} f)_\mu & \end{array}$$

Because of $x_n(id^2) \leq \|id^1\| x_n(id_\mu^*)$, $\|id^1\| \lesssim 1$, see Corollary 6.5, and

$$x_n(id^2) \gtrsim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id_{p_0,p}^{D_\mu}),$$

(see Lemma 6.3), we obtain

$$x_n(id_\mu^*) \gtrsim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id_{p_0,p}^{D_\mu}).$$

The proof is complete. ■

We need a few more results of the above type.

Lemma 6.9. *Let $0 < p_0, p < \infty$ and $0 < \epsilon < p$. Then*

$$x_n(id_\mu^*) \lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id_{p_0,p-\epsilon}^{D_\mu}). \quad (6.12)$$

Proof. We consider the following diagram

$$\begin{array}{ccc} (s_{p_0,p_0}^{t,\Omega} b)_\mu & \xrightarrow{id_\mu^*} & (s_{p,2}^{0,\Omega} f)_\mu \\ & \searrow id^2 & \nearrow id^1 \\ & (s_{p-\epsilon,p-\epsilon}^{r,\Omega} f)_\mu & \end{array}$$

Clearly, $x_n(id_\mu^*) \leq \|id^1\| x_n(id^2)$ and by Lemma 6.6 we have

$$\|id^1\| \lesssim 2^{\mu(-r+\frac{1}{p-\epsilon}-\frac{1}{p})}.$$

Further we know

$$x_n(id^2) = 2^{\mu(r-\frac{1}{p-\epsilon}-t+\frac{1}{p_0})} x_n(id_{p_0,p-\epsilon}^{D_\mu}).$$

Inserting the previous inequality in this identity we obtain (6.12). ■

Lemma 6.10. *For all $\mu \in \mathbb{N}_0$ and all $n \in \mathbb{N}$ we have*

$$x_n(id_\mu^*) \leq x_n(id^*). \quad (6.13)$$

Proof. We consider the following diagram

$$\begin{array}{ccc} s_{p_0,p_0}^{t,\Omega} b & \xrightarrow{id^*} & s_{p,2}^{0,\Omega} f \\ id^1 \uparrow & & \downarrow id^2 \\ (s_{p_0,p_0}^{t,\Omega} b)_\mu & \xrightarrow{id_\mu^*} & (s_{p,2}^{0,\Omega} f)_\mu. \end{array}$$

Here id^1 is the canonical embedding and id^2 is the canonical projection. Since $id_\mu^* = id_2 \circ id^* \circ id_1$ the property (c) of the s -numbers yields

$$x_n(id_\mu^*) \leq \|id^1\| \|id^2\| x_n(id^*) = x_n(id^*).$$

This completes the proof. ■

6.3 Weyl numbers of embeddings of sequence spaces related to spaces of dominating mixed smoothness - results

Now we are in position to deal with the Weyl numbers of $id^* : s_{p_0, p_0}^{t, \Omega} b \rightarrow s_{p, 2}^{0, \Omega} f$. We have to continue with the proof already started in (6.1)-(6.7). Therefore we need to distinguish several cases. Always the positions of p_0 and p relative to 2 are of importance.

6.3.1 The case $0 < p_0 \leq 2 \leq p < \infty$

Theorem 6.11. *Let $0 < p_0 \leq 2 \leq p < \infty$ and $t > \frac{1}{p_0} - \frac{1}{p}$. Then*

$$x_n(id^*) \asymp n^{-t+\frac{1}{2}-\frac{1}{p}} (\log n)^{(d-1)(t+\frac{1}{p}-\frac{1}{p_0})}, \quad n \geq 2.$$

Proof. *Step 1.* Estimate from below. Since $p \geq 2$, from (6.13) and (6.11) we derive

$$x_n(id^*) \gtrsim 2^{\mu(-t+\frac{1}{2}-\frac{1}{p})} x_n(id_{p_0, p}^{D_\mu}).$$

Next we choose $n = [\frac{D_\mu}{2}]$ (here $[x]$ denotes the integer part of $x \in \mathbb{R}$). Then from property (a) in Appendix A we get

$$x_n(id_{p_0, p}^{D_\mu}) \gtrsim (D_\mu)^{\frac{1}{2}-\frac{1}{p_0}} \asymp (2^\mu \mu^{d-1})^{\frac{1}{2}-\frac{1}{p_0}},$$

which implies

$$x_n(id^*) \gtrsim 2^{\mu(-t+\frac{1}{2}-\frac{1}{p})} \mu^{(d-1)(\frac{1}{2}-\frac{1}{p_0})}.$$

Because of $2^\mu \asymp \frac{n}{\log^{d-1} n}$ we conclude

$$x_n(id^*) \gtrsim n^{-t+\frac{1}{2}-\frac{1}{p}} (\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{p})}.$$

Step 2. Estimate from above. Let L, J and α as in (6.1)-(6.3). By our assumptions we obviously have

$$2^{-\alpha L} L^{(d-1)(\frac{1}{2}-\frac{1}{p_0})_+} = 2^{L(-t+\frac{1}{p_0}-\frac{1}{p})}.$$

For given J we choose $L > J$ large enough such that

$$2^{L(-t+\frac{1}{p_0}-\frac{1}{p})} \lesssim J^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} 2^{J(-t+\frac{1}{2}-\frac{1}{p})}. \quad (6.14)$$

For the sum in (6.7), we define

$$n_\mu := [D_\mu 2^{(J-\mu)\lambda}] \leq \frac{D_\mu}{2}, \quad J+1 \leq \mu \leq L,$$

where $\lambda > 1$ is at our disposal. We choose λ such that

$$t - \frac{1}{2} + \frac{1}{p} > \lambda \left(\frac{1}{p_0} - \frac{1}{2} \right) \quad (6.15)$$

which is always possible under the given restrictions. Then

$$\sum_{\mu=J+1}^L n_\mu \asymp J^{d-1} 2^J \quad (6.16)$$

follows. If $p > 2$, we choose $\epsilon > 0$ such that $2 \leq p - \epsilon$. From (6.12) we obtain

$$x_{n_\mu}(id_\mu^*) \lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_{n_\mu}(id_{p_0,p-\epsilon}^{D_\mu}).$$

If $p = 2$, then (6.9) implies

$$x_{n_\mu}(id_\mu^*) \lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{2})} x_{n_\mu}(id_{p_0,2}^{D_\mu}).$$

Employing property (a) in Appendix A we obtain

$$\begin{aligned} x_{n_\mu}(id_\mu^*) &\lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} \left(\mu^{d-1} 2^\mu 2^{(J-\mu)\lambda} \right)^{\frac{1}{2}-\frac{1}{p_0}} \\ &= \mu^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} 2^{\mu(-t+\frac{1}{2}-\frac{1}{p})} 2^{(J-\mu)\lambda(\frac{1}{2}-\frac{1}{p_0})}. \end{aligned}$$

Our special choice of λ in (6.15) yields

$$\sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) \lesssim J^{(d-1)\rho(\frac{1}{2}-\frac{1}{p_0})} 2^{J\rho(-t+\frac{1}{2}-\frac{1}{p})}. \quad (6.17)$$

Inserting (6.14) and (6.17) into (6.7) leads to

$$x_n^\rho(id^*) \lesssim J^{(d-1)\rho(\frac{1}{2}-\frac{1}{p_0})} 2^{J\rho(-t+\frac{1}{2}-\frac{1}{p})}.$$

Notice

$$n = 1 + \sum_{\mu=0}^L (n_\mu - 1) = 1 + \sum_{\mu=0}^J D_\mu + \sum_{\mu=J+1}^L ([D_\mu 2^{(J-\mu)\lambda}] - 1) \asymp J^{d-1} 2^J,$$

see (6.4), (6.5) and (6.16). Hence, our proof works for a certain subsequence $(n_J)_{J=1}^\infty$ of the natural numbers. More exactly, with

$$n_J := 1 + \sum_{\mu=0}^J D_\mu + \sum_{\mu=J+1}^L ([D_\mu 2^{(J-\mu)\lambda}] - 1), \quad J \in \mathbb{N},$$

and $L = L(J)$ chosen as the minimal admissible value in (6.14) we find

$$x_{n_J}(id^*) \lesssim J^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} 2^{J(-t+\frac{1}{2}-\frac{1}{p})}.$$

We already know

$$A J^{d-1} 2^J \leq n_J \leq B J^{d-1} 2^J, \quad J \in \mathbb{N},$$

for suitable $A, B > 0$. Without loss of generality we assume $B \in \mathbb{N}$. Then we conclude from the monotonicity of the Weyl numbers

$$x_{B J^{d-1} 2^J}(id^*) \lesssim \log(B J^{d-1} 2^J)^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} \left(\frac{B J^{d-1} 2^J}{\log^{d-1}(B J^{d-1} 2^J)} \right)^{-t+\frac{1}{2}-\frac{1}{p}}.$$

Employing one more times the monotonicity of the Weyl numbers and in addition its polynomial behaviour we can switch from the subsequence $(B J^{d-1} 2^J)_J$ to $n \in \mathbb{N}$ in this formula by possibly changing the constant behind \lesssim . This finishes our proof. \blacksquare

6.3.2 The case $2 \leq p_0 \leq p < \infty$

Theorem 6.12. *Let $2 \leq p_0 \leq p < \infty$ and $t > \frac{1}{p_0} - \frac{1}{p}$. Then*

$$x_n(id^*) \asymp n^{-t+\frac{1}{p_0}-\frac{1}{p}} (\log n)^{(d-1)(t+\frac{1}{p}-\frac{1}{p_0})}, \quad n \geq 2.$$

Proof. *Step 1.* Estimate from below. We apply the same arguments as in proof of the previous theorem. However, notice that $x_n(id_{p_0,p}^{D_\mu})$ has a different behaviour, see property (a) in Appendix A. With $n = [D_\mu/2]$ and $x_n(id_{p_0,p}^{D_\mu}) \gtrsim 1$ we conclude that

$$x_n(id^*) \gtrsim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id_{p_0,p}^{D_\mu}) \gtrsim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})},$$

see (6.13) and (6.11). Because of $2^\mu \asymp \frac{n}{\log^{d-1} n}$ this results in the estimate

$$x_n(id^*) \gtrsim n^{-t+\frac{1}{p_0}-\frac{1}{p}} (\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{p})}.$$

Step 2. Estimate from above. For $J \in \mathbb{N}$ and $\lambda \in s_{p_0,p_0}^{t,\Omega} b$ we put

$$S_J \lambda := \sum_{\mu=0}^J \sum_{|\bar{\nu}|_1=\mu} \sum_{\bar{m} \in A_{\bar{\nu}}^\Omega} \lambda_{\bar{\nu},\bar{m}} e^{\bar{\nu},\bar{m}},$$

where $\{e^{\bar{\nu},\bar{m}}, \bar{\nu} \in \mathbb{N}_0^d, \bar{m} \in A_{\bar{\nu}}^\Omega\}$ is the canonical orthonormal basis of $s_{2,2}^{0,\Omega} b$. Obviously

$$\|id^* - S_J : s_{p_0,p_0}^{t,\Omega} b \rightarrow s_{p,2}^{0,\Omega} f\| \leq \sum_{\mu=J+1}^{\infty} \|id_\mu^* : (s_{p_0,p_0}^{t,\Omega} b)_\mu \rightarrow (s_{p,2}^{0,\Omega} f)_\mu\|.$$

Using Lem. 6.6 and $(s_{p_0,p_0}^{t,\Omega} b)_\mu = (s_{p_0,p_0}^{t,\Omega} f)_\mu$ we get

$$\|id^* - S_J : s_{p_0,p_0}^{t,\Omega} b \rightarrow s_{p,2}^{0,\Omega} f\| \leq \sum_{\mu=J+1}^{\infty} 2^{-\mu(t-\frac{1}{p_0}+\frac{1}{p})} \lesssim 2^{-J(t-\frac{1}{p_0}+\frac{1}{p})}.$$

Because of $\text{rank}(S_J) \asymp 2^J J^{d-1}$ we conclude in case $n = 2^J J^{d-1}$ that

$$a_n(id^*) \lesssim 2^{-J(t-\frac{1}{p_0}+\frac{1}{p})}.$$

Since $x_n \leq a_n$ we can complete the proof of the estimate from above by arguing as at the end of the proof of Thm. 6.11. ■

6.3.3 The case $2 \leq p < p_0 \leq \infty$

Theorem 6.13. *Let $2 \leq p < p_0 \leq \infty$ and $t > \frac{1/p-1/p_0}{p_0/2-1}$. Then*

$$x_n(id^*) \asymp n^{-t+\frac{1}{p_0}-\frac{1}{p}} (\log n)^{(d-1)(t+\frac{1}{p}-\frac{1}{p_0})}, \quad n \geq 2.$$

Proof. *Step 1.* Estimate from below. Because of $p > 2$, (6.13) and (6.11) imply

$$x_n(id^*) \gtrsim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id_{p_0,p}^{D_\mu}).$$

We choose $n = \lfloor D_\mu/2 \rfloor$. Then property (b)(part(iii)) in Appendix A yields $x_n(id_{p_0,p}^{D_\mu}) \gtrsim 1$. Hence

$$x_n(id_\mu^*) \gtrsim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}.$$

Because of $2^\mu \asymp \frac{n}{\log^{d-1} n}$ this implies the desired estimate.

Step 2. Estimate from above. Since $2 \leq p < p_0$ we obtain

$$2^{-\alpha L} L^{(d-1)(\frac{1}{2}-\frac{1}{p_0})+} = 2^{-Lt} L^{(d-1)(\frac{1}{2}-\frac{1}{p_0})}.$$

For given J we choose L large enough such that

$$2^{-Lt} L^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} \leq 2^{-\gamma Jt} \quad (6.18)$$

for some $\gamma > 1$ (to be chosen later on). We define

$$n_\mu := \lfloor D_\mu 2^{(J-\mu)\beta} \rfloor \leq D_\mu, \quad J+1 \leq \mu \leq L,$$

where the parameter $\beta > 1$ will be also chosen later on. Hence

$$\sum_{\mu=J+1}^L n_\mu \asymp 2^J J^{d-1}.$$

The restriction $t > \frac{1/p-1/p_0}{p_0/2-1}$ implies

$$-t + \frac{1}{p_0} - \frac{1}{p} + \frac{1/p-1/p_0}{1-2/p_0} < 0.$$

If $p > 2$ we choose $\epsilon > 0$ such that $2 \leq p - \epsilon$ and

$$-t + \frac{1}{p_0} - \frac{1}{p} + \frac{\frac{1}{p-\epsilon} - \frac{1}{p_0}}{1 - \frac{2}{p_0}} < 0. \quad (6.19)$$

In this situation we derive from property (b)(part(i)) in Appendix A

$$x_{n_\mu}(id_{p_0,p-\epsilon}^{D_\mu}) \lesssim \left(\frac{D_\mu}{n_\mu} \right)^{\frac{1}{r}} \asymp 2^{-\frac{(J-\mu)\beta}{r}}, \quad \frac{1}{r} := \frac{\frac{1}{p-\epsilon} - \frac{1}{p_0}}{1 - \frac{2}{p_0}}.$$

The estimate (6.12) guarantees

$$\sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) \lesssim \sum_{\mu=J+1}^L 2^{\mu\rho(-t+\frac{1}{p_0}-\frac{1}{p})} x_{n_\mu}^\rho(id_{p_0,p-\epsilon}^{D_\mu}). \quad (6.20)$$

In case $p = 2$, again property (b)(part(i)) in Appendix A yields

$$x_{n_\mu}(id_{p_0,2}^{D_\mu}) \lesssim \left(\frac{D_\mu}{n_\mu} \right)^{\frac{1}{2}} \asymp 2^{-\frac{(J-\mu)\beta}{r}}, \quad \frac{1}{r} := \frac{1}{2}.$$

From (6.10) we obtain

$$\sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) \lesssim \sum_{\mu=J+1}^L 2^{\mu\rho(-t+\frac{1}{p_0}-\frac{1}{2})} x_{n_\mu}^\rho(id_{p_0,2}^{D_\mu}). \quad (6.21)$$

Now (6.20) and (6.21) yield

$$\begin{aligned} \sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) &\lesssim \sum_{\mu=J+1}^L 2^{\mu\rho(-t+\frac{1}{p_0}-\frac{1}{p})} 2^{-\frac{(J-\mu)\beta\rho}{r}} \\ &= \sum_{\mu=J+1}^L 2^{\mu\rho(-t+\frac{1}{p_0}-\frac{1}{p}+\frac{\beta}{r})} 2^{-\frac{J\beta\rho}{r}}. \end{aligned}$$

The condition (6.19) can be rewritten as

$$-t + \frac{1}{p_0} - \frac{1}{p} + \frac{1}{r} < 0.$$

Now we choose $\beta > 1$ such that $-t + \frac{1}{p_0} - \frac{1}{p} + \frac{\beta}{r} < 0$. Then

$$\sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) \lesssim 2^{J\rho(-t+\frac{1}{p_0}-\frac{1}{p}+\frac{\beta}{r})} 2^{-\frac{J\beta\rho}{r}} = 2^{J\rho(-t+\frac{1}{p_0}-\frac{1}{p})}$$

follows. Inserting this and (6.18) into (6.7) we find

$$x_n(id^*) \lesssim \left(2^{J\rho(-t+\frac{1}{p_0}-\frac{1}{p})} + 2^{-\gamma Jt\rho} \right).$$

Choosing

$$\gamma := \frac{-t + \frac{1}{p_0} - \frac{1}{p}}{-t} > 1$$

then we conclude

$$x_n(id^*) \lesssim 2^{J(-t+\frac{1}{p_0}-\frac{1}{p})}$$

and this is enough to prove the estimate from above, compare with the end of the proof of Thm. 6.11. ■

Theorem 6.14. *Let $2 \leq p < p_0 < \infty$ and $0 < t < \frac{1/p-1/p_0}{p_0/2-1}$. Then*

$$x_n(id^*) \asymp n^{-\frac{tp_0}{2}} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}, \quad n \geq 2.$$

Proof. *Step 1.* Estimate from below. From (6.9) and (6.13) we derive

$$2^{\mu(-t+\frac{1}{p_0}-\frac{1}{2})} x_n(id_{p_0,2}^{D_\mu}) \lesssim x_n(id^*).$$

Now we choose $n = [D_\mu^{\frac{2}{p_0}}]$. Then it follows from property (b)(part(ii)) in Appendix A that

$$x_n(id_{p_0,2}^{D_\mu}) \gtrsim D_\mu^{\frac{1}{2}-\frac{1}{p_0}} \gtrsim (\mu^{d-1} 2^\mu)^{\frac{1}{2}-\frac{1}{p_0}}.$$

This implies

$$x_n(id^*) \gtrsim \mu^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} 2^{-t\mu}.$$

Rewriting the right-hand side in dependence on n we obtain

$$x_n(id^*) \gtrsim n^{-\frac{tp_0}{2}} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}.$$

Step 2. Estimate from above. Since $2 \leq p < p_0$ we have

$$2^{-\alpha L} L^{(d-1)(\frac{1}{2}-\frac{1}{p_0})+} = 2^{-tL} L^{(d-1)(\frac{1}{2}-\frac{1}{p_0})}.$$

For fixed $J \in \mathbb{N}$ we choose

$$L := \left\lceil \frac{p_0}{2} J + (d-1) \left(\frac{p_0}{2} - 1 \right) \log J \right\rceil.$$

Hence

$$2^{-Lt} = 2^{-t(\lceil \frac{p_0}{2} J + (d-1)(\frac{p_0}{2}-1) \log J \rceil)} \lesssim 2^{-\frac{p_0}{2} J t} J^{(d-1)(t-\frac{tp_0}{2})}$$

and

$$L^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} = \left(\left\lceil \frac{p_0}{2} J + (d-1) \left(\frac{p_0}{2} - 1 \right) \log J \right\rceil \right)^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} \lesssim J^{(d-1)(\frac{1}{2}-\frac{1}{p_0})}.$$

This results in the estimate

$$2^{-Lt} L^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} \lesssim 2^{-\frac{p_0}{2} J t} J^{(d-1)(t-\frac{tp_0}{2}+\frac{1}{2}-\frac{1}{p_0})}. \quad (6.22)$$

We define

$$n_\mu := \left[D_\mu 2^{\{(\mu-L)\beta+J-\mu\}} \right] \leq D_\mu, \quad J+1 \leq \mu \leq L,$$

where $\beta > 0$ will be fixed later on. Consequently

$$\sum_{\mu=J+1}^L n_\mu \lesssim 2^J J^{d-1}. \quad (6.23)$$

Employing property (b)(part(i)) in Appendix A we get

$$x_{n_\mu}(id_{p_0,p}^{D_\mu}) \lesssim \left(\frac{D_\mu}{n_\mu} \right)^{\frac{1}{r}} \lesssim 2^{-\frac{(\mu-L)\beta+J-\mu}{r}}, \quad \frac{1}{r} := \frac{1/p - 1/p_0}{1 - 2/p_0}. \quad (6.24)$$

We continue by applying (6.9)

$$\begin{aligned} \sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) &\lesssim \sum_{\mu=J+1}^L \mu^{(d-1)\rho(\frac{1}{2}-\frac{1}{p})} 2^{\mu\rho(-t+\frac{1}{p_0}-\frac{1}{p})} x_{n_\mu}^\rho(id_{p_0,p}^{D_\mu}) \\ &\lesssim \sum_{\mu=J+1}^L \mu^{(d-1)\rho(\frac{1}{2}-\frac{1}{p})} 2^{\mu\rho(-t+\frac{1}{p_0}-\frac{1}{p})} 2^{-\frac{\{(\mu-L)\beta+J-\mu\}\rho}{r}} \\ &= \sum_{\mu=J+1}^L \mu^{(d-1)\rho(\frac{1}{2}-\frac{1}{p})} 2^{\mu\rho(-t+\frac{1}{p_0}-\frac{1}{p}+\frac{1}{r}-\frac{\beta}{r})} 2^{\frac{(L\beta-J)\rho}{r}}. \end{aligned}$$

Because of

$$t < \frac{1/p - 1/p_0}{p_0/2 - 1} \iff -t + \frac{1}{p_0} - \frac{1}{p} + \frac{1}{r} > 0$$

we can choose $\beta > 0$ such that $-t + \frac{1}{p_0} - \frac{1}{p} + \frac{1}{r} - \frac{\beta}{r} > 0$. Then

$$\begin{aligned} \sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) &\lesssim L^{(d-1)\rho(\frac{1}{2}-\frac{1}{p})} 2^{L\rho(-t+\frac{1}{p_0}-\frac{1}{p}+\frac{1}{r}-\frac{\beta}{r})} 2^{\frac{(L\beta-J)\rho}{r}} \\ &= L^{(d-1)\rho(\frac{1}{2}-\frac{1}{p})} 2^{L\rho(-t+\frac{1}{p_0}-\frac{1}{p}+\frac{1}{r})} 2^{-\frac{J\rho}{r}} \end{aligned} \quad (6.25)$$

follows. Inserting the definition of L we conclude

$$L^{(d-1)(\frac{1}{2}-\frac{1}{p})} \lesssim J^{(d-1)(\frac{1}{2}-\frac{1}{p})}$$

and

$$\begin{aligned} 2^{L(-t+\frac{1}{p_0}-\frac{1}{p}+\frac{1}{r})} 2^{-\frac{J}{r}} &\lesssim 2^{\left[\frac{p_0}{2}J+(d-1)(\frac{p_0}{2}-1)\log J\right] \left[-t+\frac{1}{p_0}-\frac{1}{p}+\frac{1}{r}\right]} 2^{-\frac{J}{r}} \\ &\lesssim 2^{-\frac{tp_0}{2}J} J^{(d-1)(\frac{p_0}{2}-1)(-t+\frac{1}{p_0}-\frac{1}{p}+\frac{1}{r})} \\ &= 2^{-\frac{tp_0}{2}J} J^{(d-1)(t-\frac{tp_0}{2}-\frac{1}{p_0}+\frac{1}{p})}. \end{aligned}$$

Now (6.25) yields

$$\sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) \lesssim J^{(d-1)\rho(t-\frac{tp_0}{2}-\frac{1}{p_0}+\frac{1}{2})} 2^{-\frac{tp_0}{2}J\rho}.$$

This, together with (6.22), has to be inserted into (6.7)

$$x_n(id^*) \lesssim J^{(d-1)(t-\frac{tp_0}{2}-\frac{1}{p_0}+\frac{1}{2})} 2^{-\frac{tp_0}{2}J}.$$

The same type of arguments as at the end of the proof of Thm. 6.11 complete the proof. \blacksquare

Remark 6.15. Without going into details we mention the following estimate for the limiting case $t = \frac{1/p-1/p_0}{p_0/2-1}$. For all $n \geq 2$ we have

$$n^{-\frac{tp_0}{2}}(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})} \lesssim x_n(id^*) \lesssim n^{-\frac{tp_0}{2}}(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}(\log n)^{\frac{1}{r}+\frac{1}{\rho}},$$

where r is as in (6.24) and $\rho = \min(1, p)$.

6.3.4 The case $0 < p_0, p \leq 2$

We need some preparations.

Lemma 6.16. *Let $0 < p_0, p \leq 2$ and $t > (\frac{1}{p_0} - \frac{1}{p})_+$. Then*

$$n^{-t}(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})_+} \lesssim x_n(id^*)$$

holds for all $n \geq 2$.

Proof. *Step 1.* We consider the following commutative diagram

$$\begin{array}{ccc} s_{p_0, p_0}^{t, \Omega} b & \xrightarrow{id^*} & s_{p, 2}^{0, \Omega} f \\ id^1 \uparrow & & \downarrow id^2 \\ 2^{\mu(t-\frac{1}{p_0})} \ell_{p_0}^{A_\mu} & \xrightarrow{I_\mu} & 2^{\mu(0-\frac{1}{p})} \ell_p^{A_\mu}. \end{array}$$

Here $A_\mu = |A_{\bar{\nu}}^\Omega|$ for some $\bar{\nu}$ with $|\bar{\nu}|_1 = \mu$, id^1 is the canonical embedding, whereas id^2 is the canonical projection. From property (c) of the s -numbers we derive

$$x_n(I_\mu) = x_n(id^2 \circ id^* \circ id^1) \leq \|id^1\| \|id^2\| x_n(id^*) = x_n(id^*).$$

Again the ideal property of the s -numbers guarantees

$$x_n(I_\mu) = 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id_{p_0,p}^{A_\mu}).$$

We choose $n = [A_\mu/2]$. Then property (a) in Appendix A yields

$$x_n(I_\mu) \geq 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id_{p_0,p}^{A_\mu}) \gtrsim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} 2^{\mu(\frac{1}{p}-\frac{1}{p_0})} = 2^{-\mu t} \asymp n^{-t},$$

which implies $x_n(id^*) \gtrsim n^{-t}$. This proves the lemma if $t + \frac{1}{2} - \frac{1}{p_0} \leq 0$.

Step 2. From (6.13) and (6.8) we have

$$x_n(id^*) \gtrsim \mu^{(d-1)(\frac{1}{2}-\frac{1}{p})} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id_{p_0,p}^{D_\mu}).$$

We choose $n := [D_\mu/2]$. Then property (a) in Appendix A leads to

$$x_n(id_{p_0,p}^{D_\mu}) \gtrsim D_\mu^{\frac{1}{p}-\frac{1}{p_0}} \gtrsim (\mu^{d-1} 2^\mu)^{\frac{1}{p}-\frac{1}{p_0}},$$

which implies

$$x_n(id^*) \gtrsim \mu^{(d-1)(\frac{1}{2}-\frac{1}{p})} 2^{-t\mu}.$$

Because of $2^\mu \asymp \frac{n}{\log^{d-1} n}$ this yields

$$x_n(id^*) \gtrsim n^{-t} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}.$$

The proof is complete. ■

Lemma 6.17. *If $0 < p_0, p \leq 2$ and $t > \frac{1}{p_0} - \frac{1}{2}$. Then*

$$x_n(id^*) \lesssim n^{-t} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}$$

holds for all $n \geq 2$.

Proof. The restriction $t > \frac{1}{p_0} - \frac{1}{2}$ implies the following chain of continuous embeddings

$$s_{p_0,p_0}^{t,\Omega} b \hookrightarrow s_{2,2}^{0,\Omega} f \hookrightarrow s_{p,2}^{0,\Omega} f.$$

Now we consider the commutative diagram

$$\begin{array}{ccc} s_{p_0,p_0}^{t,\Omega} b & \xrightarrow{id^*} & s_{p,2}^{0,\Omega} f \\ & \searrow id^1 \quad \nearrow id^2 & \\ & s_{2,2}^{0,\Omega} f & \end{array}$$

The ideal property of the s -numbers and Thm. 6.11 (applied with $p = 2$) yield the claim. ■

Lemma 6.18. *Let $0 < p \leq p_0 < 2$ and $0 < t < \frac{1}{p_0} - \frac{1}{2}$. Then*

$$x_n(id^*) \lesssim n^{-t}$$

holds for all $n \geq 1$.

Proof. For given $J \in \mathbb{N}$ we choose $L := J + (d-1)[\log J]$. Then

$$2^{-L\alpha} L^{(d-1)(\frac{1}{2}-\frac{1}{p_0})+} = 2^{-Lt} \asymp 2^{-tJ} J^{(d-1)(-t)}. \quad (6.26)$$

We define

$$n_\mu := [D_\mu 2^{(\mu-L)\beta+J-\mu}], \quad J+1 \leq \mu \leq L,$$

for some $\beta > 0$. Then (6.23) follows. Property (a) in Appendix A yields

$$x_{n_\mu}(id_{p_0,2}^{D_\mu}) \lesssim \left(D_\mu 2^{(\mu-L)\beta+J-\mu} \right)^{\frac{1}{2}-\frac{1}{p_0}}.$$

This, in connection with (6.8), leads to

$$\begin{aligned} \sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) &\lesssim \sum_{\mu=J+1}^L 2^{\mu\rho(-t+\frac{1}{p_0}-\frac{1}{2})} \left(D_\mu 2^{(\mu-L)\beta+J-\mu} \right)^{\rho(\frac{1}{2}-\frac{1}{p_0})} \\ &\lesssim \sum_{\mu=J+1}^L 2^{\mu\rho(-t+\frac{1}{p_0}-\frac{1}{2}+(\frac{1}{2}-\frac{1}{p_0})\beta)} \left(\mu^{(d-1)} 2^{-L\beta+J} \right)^{\rho(\frac{1}{2}-\frac{1}{p_0})}. \end{aligned}$$

Because of $t < \frac{1}{p_0} - \frac{1}{2}$ we can select $\beta > 0$ such that

$$-t + \frac{1}{p_0} - \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{p_0} \right) \beta > 0.$$

Consequently

$$\begin{aligned} \sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) &\lesssim 2^{L\rho(-t+\frac{1}{p_0}-\frac{1}{2}+(\frac{1}{2}-\frac{1}{p_0})\beta)} \left(L^{(d-1)} 2^{-L\beta+J} \right)^{\rho(\frac{1}{2}-\frac{1}{p_0})} \\ &= 2^{L\rho(-t+\frac{1}{p_0}-\frac{1}{2})} \left(L^{(d-1)} 2^J \right)^{\rho(\frac{1}{2}-\frac{1}{p_0})} \\ &\lesssim 2^{L\rho(-t+\frac{1}{p_0}-\frac{1}{2})} \left(J^{(d-1)} 2^J \right)^{\rho(\frac{1}{2}-\frac{1}{p_0})} \\ &= 2^{L\rho(-t)} 2^{L\rho(\frac{1}{p_0}-\frac{1}{2})} \left(J^{(d-1)} 2^J \right)^{\rho(\frac{1}{2}-\frac{1}{p_0})}. \end{aligned} \quad (6.27)$$

Observe

$$2^{L(\frac{1}{p_0}-\frac{1}{2})} \left(J^{(d-1)} 2^J \right)^{\frac{1}{2}-\frac{1}{p_0}} = 2^{(J+(d-1)[\log J])(\frac{1}{p_0}-\frac{1}{2})} \left(J^{(d-1)} 2^J \right)^{\frac{1}{2}-\frac{1}{p_0}} \asymp 1.$$

Replacing L by $J + (d-1)[\log J]$ in (6.27) we obtain

$$\sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) \lesssim 2^{-L\rho t} \lesssim (2^J J^{d-1})^{-\rho t}.$$

This inequality, together with (6.26), yield

$$x_{n_J}(id^*) \lesssim n_J^{-t},$$

where

$$n_J := 1 + \sum_{\mu=0}^J D_\mu + \sum_{\mu=J+1}^L \left([D_\mu 2^{(\mu-L)\beta+J-\mu}] - 1 \right), \quad J \in \mathbb{N}.$$

Now we can continue as at the end of the proof of Thm. 6.11. ■

It remains to investigate the following situation: $0 < p_0 < p < 2$ and $\frac{1}{p_0} - \frac{1}{p} < t < \frac{1}{p_0} - \frac{1}{2}$. The estimates of the Weyl numbers $x_n(id^*)$ from above will be the most complicated part within this paper.

Lemma 6.19. *Let $0 < p_0 < p < 2$ and $\frac{1}{p_0} - \frac{1}{p} < t < \frac{1}{p_0} - \frac{1}{2}$. Then*

$$x_n(id^*) \lesssim n^{-t}$$

holds for all $n \geq 1$.

Proof. *Step 1.* We need to replace the decomposition of id^* from (6.1) by a more sophisticated one:

$$id^* = \sum_{\mu=0}^J id_\mu^* + \sum_{\mu=J+1}^L id_\mu^* + \sum_{\mu=L+1}^K id_\mu^* + \sum_{\mu=K+1}^{\infty} id_\mu^* \quad \text{with } J < L < K.$$

Here J, L and K will be chosen later on. As in (6.2) this decomposition results in the estimate

$$x_n^\rho(id^*) \leq \sum_{\mu=0}^J x_{n_\mu}^\rho(id_\mu^*) + \sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) + \sum_{\mu=L+1}^K x_{n_\mu}^\rho(id_\mu^*) + \sum_{\mu=K+1}^{\infty} \|id_\mu^*\|^\rho, \quad \rho = \min(1, p), \quad (6.28)$$

where $n - 1 = \sum_{\mu=0}^K (n_\mu - 1)$. Cor. 6.5 yields

$$\|id_\mu^*\| \lesssim 2^{-\mu\alpha} \mu^{(d-1)(\frac{1}{2}-\frac{1}{p_0})_+} = 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}$$

and therefore

$$\sum_{\mu=K+1}^{\infty} \|id_\mu^*\|^\rho \lesssim 2^{K\rho(-t+\frac{1}{p_0}-\frac{1}{p})}.$$

As above we choose

$$n_\mu := D_\mu + 1, \quad \mu = 0, 1, \dots, J,$$

see (6.4). Hence

$$\sum_{\mu=0}^J n_\mu \asymp J^{d-1} 2^J \quad \text{and} \quad \sum_{\mu=0}^J x_{n_\mu}^\rho(id_\mu^*) = 0,$$

see (6.5) and (6.6). Inserting this into (6.28) we obtain

$$x_n^\rho(id^*) \lesssim \sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) + \sum_{\mu=L+1}^K x_{n_\mu}^\rho(id_\mu^*) + 2^{K\rho(-t+\frac{1}{p_0}-\frac{1}{p})}. \quad (6.29)$$

Step 2. For given J we choose K large enough such that

$$2^{K(-t+\frac{1}{p_0}-\frac{1}{p})} \leq 2^{-Jt} J^{(d-1)(-t)}.$$

Furthermore, we choose $L := J + (d-1)\lfloor \log J \rfloor$ also in dependence on J . This implies

$$2^{-Lt} \asymp 2^{-tJ} J^{(d-1)(-t)}.$$

Now we fix our remaining degrees of freedom by defining

$$n_\mu := \begin{cases} [D_\mu 2^{(\mu-L)\beta+J-\mu}] & \text{if } J+1 \leq \mu \leq L, \\ [J^{d-1} 2^J 2^{(L-\mu)\gamma}] & \text{if } L+1 \leq \mu \leq K. \end{cases}$$

Here $\beta, \gamma > 0$ will be fixed later. Since $\gamma > 0$, applying (6.23), we have

$$\sum_{\mu=J+1}^K n_\mu \asymp J^{d-1} 2^J. \quad (6.30)$$

Substep 2.1. We estimate the first sum in (6.29). Making use of the same arguments as in proof of Lemma 6.18 we find

$$\sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) \lesssim 2^{-L\rho t} \lesssim 2^{-tJ\rho} J^{-(d-1)\rho t}. \quad (6.31)$$

Substep 2.2. Now we estimate the second sum in (6.29). Therefore we consider the following splitting of n_μ , $L+1 \leq \mu \leq K$

$$n_\mu \asymp J^{d-1} 2^J 2^{(L-\mu)\gamma} = J^{d-1} 2^\mu 2^{L-\mu} 2^{-(d-1)\lfloor \log J \rfloor} 2^{(L-\mu)\gamma} = 2^\mu 2^{(L-\mu)(\gamma+1)},$$

where we used the definition of L . Observe $n_\mu \leq D_\mu/2$. The inequality (6.10) and property (a) in Appendix A lead to the estimate

$$\begin{aligned} x_{n_\mu}(id_\mu) &\lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id_{p_0,p}^{D_\mu}) \lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} (2^\mu 2^{(L-\mu)(\gamma+1)})^{\frac{1}{p}-\frac{1}{p_0}} \\ &= 2^{-\mu t} 2^{(L-\mu)(\gamma+1)(\frac{1}{p}-\frac{1}{p_0})}. \end{aligned}$$

This implies

$$\sum_{\mu=L+1}^K x_{n_\mu}^\rho(id_\mu^*) \lesssim \sum_{\mu=L+1}^K 2^{-\mu\rho t} 2^{(L-\mu)(\gamma+1)(\frac{1}{p}-\frac{1}{p_0})\rho}.$$

Choosing $\gamma > 0$ such that

$$t > (\gamma+1) \left(\frac{1}{p_0} - \frac{1}{p} \right)$$

we conclude

$$\sum_{\mu=L+1}^K x_{n_\mu}^\rho(id_\mu^*) \lesssim 2^{-Lt\rho} \asymp 2^{-tJ\rho} J^{-(d-1)t\rho}.$$

Hence, inserting the previous inequality and (6.31) into (6.29),

$$x_n(id^*) \lesssim 2^{-tJ} J^{(d-1)(-t)}$$

follows. Based on this estimate and (6.30) one can finish the proof as before. \blacksquare

As a corollary of Lem. 6.16 - Lem. 6.19 we obtain the main result of this subsection.

Theorem 6.20. *Let $0 < p_0, p \leq 2$ and $t > \left(\frac{1}{p_0} - \frac{1}{p}\right)_+$.*

(i) *If $t > \frac{1}{p_0} - \frac{1}{2}$, then*

$$x_n(id^*) \asymp n^{-t}(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}$$

holds for all $n \geq 2$.

(ii) *If $t < \frac{1}{p_0} - \frac{1}{2}$, then*

$$x_n(id^*) \asymp n^{-t}$$

holds for all $n \geq 1$.

Remark 6.21. Again we comment on the limiting situation $t = \frac{1}{p_0} - \frac{1}{2}$. For $0 < p_0, p < 2$, $\rho := \min(1, p)$ and $t = \frac{1}{p_0} - \frac{1}{2}$ it follows

$$n^{-t} \lesssim x_n(id^*) \lesssim n^{-t}(\log \log n)^{t+\frac{1}{\rho}}, \quad n \geq 3.$$

This is the only limiting case where the gap is of order $\log \log n$ to some power. For that reason we give a few more details. In principal we argue as in Lemma 6.18. For given $J \in \mathbb{N}$, $J \geq 4$, we choose $L := J + (d-1) \lfloor \log J \rfloor$ as above. Next we define

$$n_\mu := \left\lceil \frac{2^J J^{d-1}}{\log J} \right\rceil, \quad J+1 \leq \mu \leq L.$$

Then

$$\sum_{\mu=J+1}^L n_\mu \asymp 2^J J^{d-1}$$

and

$$x_{n_\mu}(id_{p_0,2}^{D_\mu}) \lesssim \left(\frac{2^J J^{d-1}}{\log J} \right)^{\frac{1}{2}-\frac{1}{p_0}}$$

follow, see property (a) in Appendix A. Applying (6.8) we find

$$\begin{aligned} \sum_{\mu=J+1}^L x_{n_\mu}^\rho(id_\mu^*) &\lesssim \sum_{\mu=J+1}^L 2^{\mu\rho(-t+\frac{1}{p_0}-\frac{1}{2})} \left(\frac{2^J J^{d-1}}{\log J} \right)^{\rho(\frac{1}{2}-\frac{1}{p_0})} \\ &\lesssim (2^J J^{d-1})^{-\rho t} (\log J)^{1+\rho t}. \end{aligned}$$

As in Lemma 6.18 this proves the claim.

6.3.5 The case $0 < p \leq 2 < p_0 \leq \infty$

This is the last case we have to consider.

Theorem 6.22. *Let $0 < p \leq 2 < p_0 \leq \infty$ and $t > \frac{1}{p_0}$. Then*

$$x_n(id^*) \asymp n^{-t+\frac{1}{p_0}-\frac{1}{2}} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}$$

holds for all $n \geq 2$.

Proof. *Step 1.* Estimate from below. Since $p \leq 2$, from (6.13) and (6.8) we derive

$$x_n(id^*) \gtrsim \mu^{(d-1)(\frac{1}{2}-\frac{1}{p})} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id_{p_0,p}^{D_\mu}).$$

We choose $n := [D_\mu/2]$ and obtain from property (c)(part(i)) in Appendix A that

$$x_n(id_{p_0,p}^{D_\mu}) \gtrsim (D_\mu)^{\frac{1}{p}-\frac{1}{2}} \gtrsim (\mu^{d-1} 2^\mu)^{\frac{1}{p}-\frac{1}{2}}.$$

This implies

$$x_n(id^*) \gtrsim 2^{\mu(-t-\frac{1}{2}+\frac{1}{p_0})}.$$

Using $2^\mu \asymp \frac{n}{\log^{d-1} n}$ we conclude

$$x_n(id^*) \gtrsim n^{-t-\frac{1}{2}+\frac{1}{p_0}} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}.$$

Step 2. Estimate from above. We consider the commutative diagram

$$\begin{array}{ccc} s_{p_0,p_0}^{t,\Omega} b & \xrightarrow{id^*} & s_{p,2}^{0,\Omega} f \\ & \searrow id^1 & \nearrow id^2 \\ & s_{2,2}^{0,\Omega} f & \end{array}$$

From $p < 2$ we derive $s_{2,2}^{0,\Omega} f \hookrightarrow s_{p,2}^{0,\Omega} f$ which implies $\|id^2\| < \infty$. The ideal property of the s -numbers in combination with Thm. 6.13 yield

$$x_n(id^*) \lesssim n^{-t+\frac{1}{p_0}-\frac{1}{2}} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}$$

if $t > \frac{1/2-1/p_0}{p_0/2-1} = \frac{1}{p_0}$. ■

Theorem 6.23. *Let $0 < p \leq 2 < p_0 < \infty$ and $0 < t < \frac{1}{p_0}$. Then*

$$x_n(id^*) \asymp n^{-\frac{tp_0}{2}} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}$$

holds for all $n \geq 2$.

Proof. *Step 1.* Estimate from below. Since $p \leq 2$, from (6.13) and (6.8) we obtain

$$x_n(id^*) \gtrsim \mu^{(d-1)(\frac{1}{2}-\frac{1}{p})} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id_{p_0,p}^{D_\mu}).$$

With $n := [D_\mu^{\frac{2}{p_0}}]$ property (c)(part(ii)) in Appendix A yields

$$x_n(id_{p_0,p}^{D_\mu}) \gtrsim D_\mu^{\frac{1}{p}-\frac{1}{p_0}} \gtrsim (\mu^{d-1} 2^\mu)^{\frac{1}{p}-\frac{1}{p_0}}.$$

Hence

$$x_n(id^*) \gtrsim \mu^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} 2^{-t\mu}.$$

Since $2^\mu \asymp \frac{n^{\frac{p_0}{2}}}{\log^{d-1}(n^{\frac{p_0}{2}})}$ we conclude

$$x_n(id^*) \gtrsim n^{-\frac{tp_0}{2}} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}.$$

Step 2. Estimate from above. Again we consider the commutative diagram

$$\begin{array}{ccc} s_{p_0,p_0}^{t,\Omega} b & \xrightarrow{id^*} & s_{p,2}^{0,\Omega} f \\ & \searrow id^1 & \nearrow id^2 \\ & s_{2,2}^{0,\Omega} f & \end{array}$$

In addition we know

$$x_n(id^1) \asymp n^{-\frac{tp_0}{2}} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}, \quad n \geq 2.$$

if $2 < p_0 < \infty$ and $0 < t < \frac{1}{p_0}$, see Thm. 6.14. Now the ideal property of the s -numbers yields

$$x_n(id^*) \lesssim n^{-\frac{tp_0}{2}} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}, \quad n \geq 2$$

if $0 < t < \frac{1}{p_0}$. ■

Remark 6.24. In the limiting situation $t = \frac{1}{p_0} > 0$ we have

$$n^{-\frac{1}{2}} (\log n)^{\frac{(d-1)}{2}} \lesssim x_n(id^*) \lesssim n^{-\frac{1}{2}} (\log n)^{\frac{(d-1)}{2}} (\log n)^{\frac{1}{2}+\frac{1}{\rho}}$$

for all $n \geq 2$. Here $\rho = \min(1, p)$.

7 Proofs

Here we will give proofs of the assertions in Section 3. For better readability we continue to work with (p_0, p) instead of (p_1, p_2) .

7.1 Proof of the main Theorem 3.1

The heart of the matter is the following in principal well-known lemma.

Lemma 7.1. *Let $0 < p_0 \leq \infty$, $0 < p < \infty$ and $t \in \mathbb{R}$. Then*

$$x_n(id^* : s_{p_0, p_0}^{t, \Omega} b \rightarrow s_{p, 2}^{0, \Omega} f) \asymp x_n(id : S_{p_0, p_0}^t B(\Omega) \rightarrow S_{p, 2}^0 F(\Omega))$$

holds for all $n \in \mathbb{N}$.

Proof. *Step 1.* Let $0 < p_0 < \infty$. Let $E : B_{p_0, p_0}^t(0, 1) \rightarrow B_{p_0, p_0}^t(\mathbb{R})$ denote a linear and continuous extension operator. For existence of those operators we refer, e.g., to [57, 3.3.4] or [45]. Without loss of generality we may assume that

$$\text{supp } Ef \subset \bigcup_{\bar{\nu} \in \mathbb{N}_0^d, \bar{m} \in A_{\bar{\nu}}^\Omega} \text{supp } \Psi_{\bar{\nu}, \bar{m}},$$

see Section 5, for all f . Then the d -fold tensor product operator

$$\mathcal{E}_d := E \otimes \dots \otimes E$$

maps the tensor product space $S_{p_0, p_0}^t B((0, 1)^d) = B_{p_0, p_0}^t(0, 1) \otimes_{\gamma_{p_0}} \dots \otimes_{\gamma_{p_0}} B_{p_0, p_0}^t(0, 1)$ into the tensor product space $S_{p_0, p_0}^t B(\mathbb{R}^d) = B_{p_0, p_0}^t(\mathbb{R}) \otimes_{\gamma_{p_0}} \dots \otimes_{\gamma_{p_0}} B_{p_0, p_0}^t(\mathbb{R})$, see [48], and is again a linear and continuous extension operator. This follows from the fact that γ_{p_0} is an uniform quasi-norm. Hence $\mathcal{E}_d \in \mathcal{L}(S_{p_0, p_0}^t B(\Omega), S_{p_0, p_0}^t B(\mathbb{R}^d))$.

Step 2. Let $p_0 = \infty$. We discussed extension operators in this case in Subsection 3.4. Now we can argue as in Step 1.

Step 3. We follow [61] and consider the commutative diagram

$$\begin{array}{ccccc} S_{p_0, p_0}^t B(\Omega) & \xrightarrow{\mathcal{E}_d} & S_{p_0, p_0}^t B(\mathbb{R}^d) & \xrightarrow{\mathcal{W}} & s_{p_0, p_0}^{t, \Omega} b \\ \text{\scriptsize } id \downarrow & & & & \downarrow \text{\scriptsize } id^* \\ S_{p, 2}^0 F(\Omega) & \xleftarrow{R_\Omega} & S_{p, 2}^0 F(\mathbb{R}^d) & \xleftarrow{\mathcal{W}^*} & s_{p, 2}^{0, \Omega} f \end{array}$$

The mapping \mathcal{W} is defined as

$$\mathcal{W}f := \left(2^{|\bar{\nu}|_1} \langle f, \Psi_{\bar{\nu}, \bar{k}} \rangle \right)_{\bar{\nu} \in \mathbb{N}_0^d, \bar{k} \in A_{\bar{\nu}}^\Omega}.$$

Furthermore, \mathcal{W}^* is defined as

$$\mathcal{W}^* \lambda := \sum_{\bar{\nu} \in \mathbb{N}_0^d} \sum_{\bar{k} \in A_{\bar{\nu}}^\Omega} \lambda_{\bar{\nu}, \bar{k}} \Psi_{\bar{\nu}, \bar{k}}$$

and R_Ω means the restriction to Ω . The boundedness of $\mathcal{E}_d, \mathcal{W}, \mathcal{W}^*, R_\Omega$ and the ideal property of the s -numbers yield $x_n(id) \lesssim x_n(id^*)$. A similar argument with a slightly modified diagram yields $x_n(id^*) \lesssim x_n(id)$ as well. \blacksquare

Next we need to recall an adapted Littlewood-Paley assertion, see Nikol'skij [31, 1.5.6].

Lemma 7.2. *Let $1 < p < \infty$. Then*

$$S_{p,2}^0 F(\mathbb{R}^d) = L_p(\mathbb{R}^d) \quad \text{and} \quad S_{p,2}^0 F(\Omega) = L_p(\Omega)$$

in the sense of equivalent norms.

Proof of Thm. 3.1. Lemma 7.1 and Lemma 7.2 allow to carry over the results obtained in Section 6 to the level of function spaces. Theorem 3.1 becomes a consequence of Theorems 6.11 - 6.14, Theorem 6.20 and Theorems 6.22, 6.23. \blacksquare

7.2 Proofs of the results in Subsections 3.2

Recall that $s_{\infty,q}^{t,\Omega} a$ or $(s_{\infty,q}^{t,\Omega} a)_\mu$ has to be interpreted as $s_{\infty,q}^{t,\Omega} b$ and $(s_{\infty,q}^{t,\Omega} b)_\mu$.

Lemma 7.3. *Let $t, r \in \mathbb{R}$ and $0 < p, q, p_0, q_0 \leq \infty$. Then*

$$x_n(id^1 : s_{p_0,q_0}^{t,\Omega} a \rightarrow s_{p,q}^{r,\Omega} a) \asymp x_n(id^2 : s_{p_0,q_0}^{t-r,\Omega} a \rightarrow s_{p,q}^{0,\Omega} a), \quad n \in \mathbb{N}.$$

Proof. We consider the commutative diagram

$$\begin{array}{ccc} s_{p_0,q_0}^{t,\Omega} a & \xrightarrow{id^1} & s_{p,q}^{r,\Omega} a \\ J_r \downarrow & & \uparrow J_{-r} \\ s_{p_0,q_0}^{t-r,\Omega} a & \xrightarrow{id^2} & s_{p,q}^{0,\Omega} a. \end{array}$$

Here J_r is the isomorphism defined in (5.1). Hence $x_n(id^1) \lesssim x_n(id^2)$. But

$$\begin{array}{ccc} s_{p_0,q_0}^{t-r,\Omega} a & \xrightarrow{id^2} & s_{p,q}^{0,\Omega} a \\ J_{-r} \downarrow & & \uparrow J_r \\ s_{p_0,q_0}^{t,\Omega} a & \xrightarrow{id^1} & s_{p,q}^{r,\Omega} a \end{array}$$

yields $x_n(id^2) \lesssim x_n(id^1)$ as well. The proof is complete. \blacksquare

Proof of Theorem 3.4. *Step 1.* Estimate from above. Under the given restrictions there always exists some $r > \frac{1}{2}$ such that $t > r + \left(\frac{1}{p_0} - \frac{1}{2}\right)_+$. We consider the commutative diagram

$$\begin{array}{ccc} S_{p_0,p_0}^t B((0,1)^d) & \xrightarrow{id_1} & L_\infty((0,1)^d) \\ & \searrow id_2 \quad \nearrow id_3 & \\ & S_{2,2}^r B((0,1)^d) & \end{array}$$

The multiplicativity of the Weyl numbers yields

$$x_{2n-1}(id_1) \leq x_n(id_2) x_n(id_3).$$

From Lemmas 7.2 and 7.3 we have

$$x_n(id_2) \asymp x_n(id : S_{p_0, p_0}^{t-r} B((0, 1)^d) \rightarrow L_2((0, 1)^d)). \quad (7.1)$$

Prop. 3.2, Thm. 3.1 and (7.1) lead to

$$x_{2n-1}(id_1) \lesssim \frac{(\log n)^{(d-1)r}}{n^{r-\frac{1}{2}}} \begin{cases} \frac{(\log n)^{(d-1)(t-r-\frac{1}{p_0}+\frac{1}{2})}}{n^{t-r}} & \text{if } 0 < p_0 \leq 2, t-r > \frac{1}{p_0} - \frac{1}{2}, \\ \frac{(\log n)^{(d-1)(t-r-\frac{1}{p_0}+\frac{1}{2})}}{n^{t-r-\frac{1}{p_0}+\frac{1}{2}}} & \text{if } 2 \leq p_0 \leq \infty, t-r > \frac{1}{p_0}. \end{cases}$$

Finally, the monotonicity of the Weyl numbers yields the claim for all $n \geq 2$.

Step 2. Estimate from below. The claim will follow from the next proposition.

Proposition 7.4. *Let $t > \frac{1}{p_0}$. As estimates from below we get*

$$x_n(id : S_{p_0, p_0}^t B((0, 1)^d) \rightarrow L_\infty((0, 1)^d)) \gtrsim \begin{cases} \frac{(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}}{n^{t-\frac{1}{2}}} & \text{if } 0 < p_0 \leq 2, \\ \frac{(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}}{n^{t-\frac{1}{p_0}}} & \text{if } 2 \leq p_0 \leq \infty, \end{cases}$$

for all $n \geq 2$.

Proof. Again we shall use the multiplicativity of the Weyl numbers, but this time in connection with its relation to the 2-summing norm [33, Lemma 8]. Let us recall this notion.

An operator $T \in \mathcal{L}(X, Y)$ is said to be *absolutely 2-summing* if there is a constant $C > 0$ such that for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$ the inequality

$$\left(\sum_{j=1}^n \|Tx_j\|^2 \right)^{1/2} \leq C \sup_{x^* \in X^*, \|x^*\| \leq 1} \left(\sum_{j=1}^n |\langle x_j, x^* \rangle|^2 \right)^{1/2} \quad (7.2)$$

holds (see [32, Chapter 17]). The norm $\pi_2(T)$ is given by the infimum of all $C > 0$ satisfying (7.2). X^* refers to the dual space of X . Pietsch [33] proved the inequality

$$n^{1/2} x_n(S) \leq \pi_2(S), \quad n \in \mathbb{N},$$

for any linear operator S . Using this inequality with respect $S = id$ we conclude

$$\begin{aligned} & x_{2n-1}(id : S_{p_0, p_0}^t B((0, 1)^d) \rightarrow L_2((0, 1)^d)) \\ & \leq x_n(id : S_{p_0, p_0}^t B((0, 1)^d) \rightarrow L_\infty((0, 1)^d)) x_n(id : L_\infty((0, 1)^d) \rightarrow L_2((0, 1)^d)) \\ & \leq x_n(id : S_{p_0, p_0}^t B((0, 1)^d) \rightarrow L_\infty((0, 1)^d)) n^{-1/2} \pi_2(id : L_\infty((0, 1)^d) \rightarrow L_2((0, 1)^d)) \\ & = x_n(id : S_{p_0, p_0}^t B((0, 1)^d) \rightarrow L_\infty((0, 1)^d)) n^{-1/2}; \end{aligned}$$

where in the last equality we have used that

$$\pi_2(id : L_\infty((0, 1)^d) \longrightarrow L_2((0, 1)^d)) = \|id : L_\infty((0, 1)^d) \longrightarrow L_2((0, 1)^d)\| = 1,$$

see [35, Example 1.3.9]). Since

$$n^{\frac{1}{2}} x_{2n-1}(id : S_{p_0, p_0}^t B((0, 1)^d) \rightarrow L_2((0, 1)^d)) \asymp \begin{cases} \frac{(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}}{n^{t-\frac{1}{2}}} & \text{if } 0 < p_0 \leq 2, t > \frac{1}{p_0} - \frac{1}{2}, \\ \frac{(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}}{n^{t-\frac{1}{p_0}}} & \text{if } 2 \leq p_0 \leq \infty, t > \frac{1}{p_0}, \end{cases}$$

see Thm. 6.20, Thm. 6.13, this proves the claimed estimate from below. \blacksquare

7.3 Proof of the results in Subsection 3.3

As a preparation we need the following counterpart of the classical result $F_{1,2}^0(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d)$ in the dominating mixed situation. The following proof we learned from Dachun Yang and Wen Yuan [64].

Lemma 7.5. *We have*

$$S_{1,2}^0 F(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d).$$

Proof. Let $f \in S_{1,2}^0 F(\mathbb{R}^d)$. We may assume that f is a Schwartz function, due to the density of $\mathcal{S}(\mathbb{R}^d)$ in $S_{1,2}^0 F(\mathbb{R}^d)$. Let $(\varphi_{\bar{j}})_{\bar{j}}$ be the smooth dyadic decomposition of unity defined in (9.2). Let $\phi_0, \phi \in C_0^\infty(\mathbb{R})$ be functions s.t.

$$\begin{aligned} \phi_0(t) &= 1 & \text{on } \text{supp } \varphi_0 \\ \phi(t) &= 1 & \text{on } \text{supp } \varphi_1. \end{aligned}$$

We put $\phi_j(t) := \phi(2^{-j+1}t)$, $j \in \mathbb{N}$, and

$$\phi_{\bar{j}} := \phi_{j_1} \otimes \dots \otimes \phi_{j_d}, \quad \bar{j} = (j_1, \dots, j_d) \in \mathbb{N}_0^d.$$

It follows

$$\sum_{\bar{j} \in \mathbb{N}_0^d} \varphi_{\bar{j}}(x) \cdot \phi_{\bar{j}}(x) = 1 \quad \text{for all } x \in \mathbb{R}^d,$$

see (9.1) and (9.2). This implies

$$f = \sum_{\bar{j} \in \mathbb{N}_0^d} \mathcal{F}^{-1}[\varphi_{\bar{j}}(\xi) \phi_{\bar{j}}(\xi) \mathcal{F}f(\xi)] \quad (\text{convergence in } S'(\mathbb{R}^d)).$$

Let $g \in L_\infty(\mathbb{R}^d)$. Hölder's inequality yields

$$\begin{aligned} |\langle f, g \rangle| &\leq \sum_{\bar{j} \in \mathbb{N}_0^d} |\langle \mathcal{F}^{-1}[\varphi_{\bar{j}} \mathcal{F}f], \mathcal{F}^{-1}[\phi_{\bar{j}} \mathcal{F}g] \rangle| \\ &\leq \left\| \left(\sum_{\bar{j} \in \mathbb{N}_0^d} |\mathcal{F}^{-1}[\varphi_{\bar{j}} \mathcal{F}f]|^2 \right)^{\frac{1}{2}} \right\|_{L_1(\mathbb{R}^d)} \left\| \left(\sum_{\bar{j} \in \mathbb{N}_0^d} |\mathcal{F}^{-1}[\phi_{\bar{j}} \mathcal{F}g]|^2 \right)^{\frac{1}{2}} \right\|_{L_\infty(\mathbb{R}^d)}. \end{aligned}$$

Next we are going to use the tensor product structure of $\mathcal{F}^{-1}\phi_{\bar{j}}$ and the fact that $\mathcal{F}^{-1}\phi_{j_l}$, $l = 1, \dots, d$, are Schwartz functions. For any $M > 0$, we have

$$\begin{aligned} |\mathcal{F}^{-1}[\phi_{\bar{j}}\mathcal{F}g](x)| &\lesssim \left| \int_{\mathbb{R}^d} g(y) (\mathcal{F}^{-1}\phi_{\bar{j}})(x-y) dy \right| \\ &\lesssim \|g\|_{L_\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{2^{|\bar{j}|_1}}{\prod_{l=1}^d (1+2^{j_l}|x_l-y_l|)^{(1+M)}} dy \\ &= \|g\|_{L_\infty(\mathbb{R}^d)} \prod_{l=1}^d \int_{\mathbb{R}} \frac{2^{j_l}}{(1+2^{j_l}|x_l-y_l|)^{(1+M)}} dy. \end{aligned}$$

Some elementary calculations yield

$$\int_{\mathbb{R}} \frac{2^{j_l}}{(1+2^{j_l}|x_l-y_l|)^{(1+M)}} dy \lesssim 2^{-j_l M}$$

with constants independent of j_l . Inserting this in our previous estimate we obtain

$$|\mathcal{F}^{-1}[\phi_{\bar{j}}\mathcal{F}g](x)| \lesssim \|g\|_{L_\infty(\mathbb{R}^d)} 2^{-|\bar{j}|_1 M}.$$

Hence

$$\begin{aligned} \left\| \left(\sum_{\bar{j} \in \mathbb{N}_0^d} |\mathcal{F}^{-1}[\phi_{\bar{j}}\mathcal{F}g](x)|^2 \right)^{\frac{1}{2}} \right\|_{L_\infty} &\lesssim \|g\|_{L_\infty(\mathbb{R}^d)} \sum_{\bar{j} \in \mathbb{N}_0^d} 2^{-|\bar{j}|_1 M} \\ &\lesssim \|g\|_{L_\infty(\mathbb{R}^d)} \sum_{\mu=0}^{\infty} \sum_{|\bar{j}|_1=\mu} 2^{-|\bar{j}|_1 M} \\ &\lesssim \|g\|_{L_\infty(\mathbb{R}^d)}. \end{aligned}$$

Therefore, we obtain

$$\|f\|_{L_1(\mathbb{R}^d)} = \sup_{\|g\|_{L_\infty(\mathbb{R}^d)}=1} |\langle f, g \rangle| \lesssim \left\| \left(\sum_{\bar{j} \in \mathbb{N}_0^d} |\mathcal{F}^{-1}[\phi_{\bar{j}}\mathcal{F}f]|^2 \right)^{\frac{1}{2}} \right\|_{L_1(\mathbb{R}^d)}.$$

That completes our proof. ■

Proof of Theorem 3.7. *Step 1.* Estimate from above. From the chain of embeddings

$$S_{p_0, p_0}^t B((0, 1)^d) \hookrightarrow S_{1, 2}^0 F((0, 1)^d) \hookrightarrow L_1((0, 1)^d),$$

together with Lem. 7.1, Thm. 6.20, Thm. 6.22, Thm. 6.23 and the abstract properties of Weyl numbers, see Section 4, we derive the upper bound.

Step 2. We prove the lower bound for the case $p_0 < 2$ and $t < \frac{1}{p_0} - \frac{1}{2}$. First we note that, under the condition $t > \max(0, \frac{1}{p_0} - 1)$, the chain of embeddings holds true

$$S_{p_0, p_0}^t B((0, 1)^d) \hookrightarrow L_1((0, 1)^d) \hookrightarrow S_{1, \infty}^0 B((0, 1)^d).$$

Then the ideal property of the s -numbers yields

$$x_n(id : S_{p_0, p_0}^t B((0, 1)^d) \rightarrow S_{1, \infty}^0 B((0, 1)^d)) \lesssim x_n(S_{p_0, p_0}^t B((0, 1)^d) \rightarrow L_1((0, 1)^d)). \quad (7.3)$$

Next we consider the commutative diagram

$$\begin{array}{ccc} B_{p_0, p_0}^{t+r}(0, 1) & \xrightarrow{id_1} & B_{1, \infty}^r(0, 1) \\ \text{Ext} \downarrow & & \uparrow \text{Tr} \\ S_{p_0, p_0}^{t+r}B((0, 1)^d) & \xrightarrow{id} & S_{1, \infty}^rB((0, 1)^d) \end{array}$$

Here the linear operators Ext and Tr are defined as follows. For $g \in B_{p_0, p_0}^{t+r}(0, 1)$, we put

$$(\text{Ext}g)(x_1, \dots, x_d) = g(x_1), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

In case of $f \in S_{1, \infty}^rB((0, 1)^d)$ we define

$$(\text{Tr}f)(x_1) = f(x_1, 0, \dots, 0), \quad x_1 \in \mathbb{R}.$$

Note that the condition $r > 1$ guarantees that the operator Tr is well defined, see [47, Thm. 2.4.2]. Furthermore, Ext maps $B_{p_0, p_0}^{t+r}(0, 1)$ continuously into $S_{p_0, p_0}^{t+r}B((0, 1)^d)$. This follows from the fact that $\|\cdot\|_{S_{p_0, p_0}^{t+r}B((0, 1)^d)}$ is a cross-quasi-norm, see the formula in Rem. 9.4(i). Hence $id_1 = \text{Tr} \circ id \circ \text{Ext}$ and

$$\begin{aligned} x_n(id_1 : B_{p_0, p_0}^{t+r}(0, 1) &\rightarrow B_{1, \infty}^r(0, 1)) \\ &\lesssim x_n(id : S_{p_0, p_0}^{t+r}B((0, 1)^d) \rightarrow S_{1, \infty}^rB((0, 1)^d)). \end{aligned} \quad (7.4)$$

Making use of a lifting argument, see Lem. 7.3, we conclude that

$$\begin{aligned} x_n(id : S_{p_0, p_0}^{t+r}B((0, 1)^d) &\rightarrow S_{1, \infty}^rB((0, 1)^d)) \\ &\asymp x_n(id : S_{p_0, p_0}^tB((0, 1)^d) \rightarrow S_{1, \infty}^0B((0, 1)^d)). \end{aligned} \quad (7.5)$$

The lower bound is now obtained from (7.3), (7.4), (7.5) and

$$x_n(id_1 : B_{p_0, p_0}^{t+r}(0, 1) \rightarrow B_{1, \infty}^r(0, 1)) \asymp n^{-t}, \quad n \in \mathbb{N},$$

if $0 < p_0 \leq 2$ and $t > \max(0, \frac{1}{p_0} - 1)$, see Lubitz [29] and Caetano [10].

Step 3. We prove that

$$x_n(id : S_{p_0, p_0}^tB((0, 1)^d) \rightarrow L_1((0, 1)^d)) \gtrsim n^{-t}(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}$$

if $p_0 \leq 2$ and $t > \frac{1}{p_0} - \frac{1}{2}$. There always exists a pair (θ, p) such that

$$0 < \theta < 1, \quad 1 < p < 2$$

and

$$\|f\|_{L_p((0, 1)^d)} \leq \|f\|_{L_1((0, 1)^d)}^{1-\theta} \|f\|_{L_2((0, 1)^d)}^\theta \quad \text{for all } f \in L_2((0, 1)^d).$$

Next we employ the interpolation property of the Weyl numbers, see Thm. 4.2, and obtain

$$\begin{aligned} x_{2n-1}(id : S_{p_0, p_0}^tB((0, 1)^d) &\rightarrow L_p((0, 1)^d)) \lesssim \\ x_n^{1-\theta}(id : S_{p_0, p_0}^tB((0, 1)^d) &\rightarrow L_1((0, 1)^d)) x_n^\theta(id : S_{p_0, p_0}^tB((0, 1)^d) \rightarrow L_2((0, 1)^d)). \end{aligned}$$

Note that $0 < p_0 \leq 2$ and $t > \frac{1}{p_0} - \frac{1}{2}$ imply

$$\begin{aligned} x_n(id : S_{p_0, p_0}^t B((0, 1)^d) \rightarrow L_2((0, 1)^d)) &\asymp x_n(id : S_{p_0, p_0}^t B((0, 1)^d) \rightarrow L_p((0, 1)^d)) \\ &\asymp n^{-t} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}, \end{aligned}$$

see Thm. 3.1. This leads to

$$x_n(id : S_{p_0, p_0}^t B((0, 1)^d) \rightarrow L_1((0, 1)^d)) \gtrsim n^{-t} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}.$$

The lower bounds in the remaining cases can be proved similarly. ■

7.4 Proofs of the results in Subsection 3.4

Proof of Theorem 3.11. Define $id^* : s_{p_0, p_0}^{t, \Omega} b \rightarrow s_{\infty, \infty}^{0, \Omega} b$ and $id_\mu^* : (s_{p_0, p_0}^{t, \Omega} b)_\mu \rightarrow (s_{\infty, \infty}^{0, \Omega} b)_\mu$.

Cor. 6.5 yields

$$\|id_\mu^*\| \lesssim 2^{\mu(\frac{1}{p_0}-t)}. \quad (7.6)$$

Arguing as in proof of Prop. 6.7 one can establish the following.

Lemma 7.6. *Let $0 < p_0 \leq \infty$ and $t \in \mathbb{R}$. Then*

$$x_n(id_\mu^*) \asymp x_n(id_\mu^{**} : 2^{\mu(t-\frac{1}{p_0})} \ell_{p_0}^{D_\mu} \rightarrow \ell_\infty^{D_\mu}) \asymp 2^{\mu(-t+\frac{1}{p_0})} x_n(id_{p_0, \infty}^{D_\mu})$$

for all $n \in \mathbb{N}$.

Property (a) in Appendix A yields

$$x_n(id_{p_0, \infty}^{D_\mu}) \asymp \begin{cases} 1 & \text{if } 2 \leq p_0 \leq \infty, \\ n^{\frac{1}{2}-\frac{1}{p_0}} & \text{if } 0 < p_0 \leq 2, \end{cases}$$

if $2n \leq D_\mu$. Now we may follow the proof of Thm. 6.11. This results in the following useful statement.

Theorem 7.7. (i) *Let $0 < p_0 \leq 2$ and $t > \frac{1}{p_0}$. Then*

$$x_n(id^* : s_{p_0, p_0}^{t, \Omega} b \rightarrow s_{\infty, \infty}^{0, \Omega} b) \asymp n^{-t+\frac{1}{2}} (\log n)^{(d-1)(t-\frac{1}{p_0})}, \quad n \geq 2.$$

(ii) *Let $2 \leq p_0 \leq \infty$ and $t > \frac{1}{p_0}$. Then*

$$x_n(id^* : s_{p_0, p_0}^{t, \Omega} b \rightarrow s_{\infty, \infty}^{0, \Omega} b) \asymp n^{-t+\frac{1}{p_0}} (\log n)^{(d-1)(t-\frac{1}{p_0})}, \quad n \geq 2.$$

By making use of a lifting argument, see Lemma 7.3, and the counterpart of Lemma 7.1 for this situation, i.e.,

$$x_n(id^* : s_{p_0, p_0}^{t, \Omega} b \rightarrow s_{\infty, \infty}^{0, \Omega} b) \asymp x_n(id : S_{p_0, p_0}^t B(\Omega) \rightarrow S_{\infty, \infty}^0 B(\Omega)), \quad n \in \mathbb{N},$$

we immediately get the following corollary.

Corollary 7.8. *Let $s \in \mathbb{R}$.*

(i) *Let $0 < p_0 \leq 2$ and $t > s + \frac{1}{p_0}$. Then*

$$x_n(id : S_{p_0, p_0}^t B((0, 1)^d) \rightarrow S_{\infty, \infty}^s B((0, 1)^d)) \asymp n^{-t+s+\frac{1}{2}} (\log n)^{(d-1)(t-s-\frac{1}{p_0})}, \quad n \geq 2.$$

(ii) *Let $2 \leq p_0 \leq \infty$ and $t > s + \frac{1}{p_0}$. Then*

$$x_n(id : S_{p_0, p_0}^t B((0, 1)^d) \rightarrow S_{\infty, \infty}^s B((0, 1)^d)) \asymp n^{-t+s+\frac{1}{p_0}} (\log n)^{(d-1)(t-s-\frac{1}{p_0})}, \quad n \geq 2.$$

Now Thm. 3.11 follows from $\mathcal{Z}_{\text{mix}}^s((0, 1)^d) = S_{\infty, \infty}^s B((0, 1)^d)$, see Lemma 3.10. \blacksquare

Proof of Theorem 3.13. The lower estimate in the case of high smoothness is a direct consequence of $x_n \leq a_n$ and Theorem 3.11.

Step 1. We prove the upper bound of $a_n(id : S_{p_0, p_0}^t B((0, 1)^d) \rightarrow S_{\infty, \infty}^0 B((0, 1)^d))$ in case $p_0 > 1$. First, recall

$$a_n(id_{p_0, \infty}^{D_\mu}) \asymp \begin{cases} 1 & \text{if } 2 \leq p_0 \leq \infty, \\ \min(1, D_\mu^{1-\frac{1}{p_0}} n^{-\frac{1}{2}}) & \text{if } 1 < p_0 < 2, \end{cases}$$

if $2n \leq D_\mu$, see [23, 62]. To avoid nasty calculations by checking this behaviour for $p_0 \geq 2$ one may use the elementary chain of inequalities

$$x_n(id_{p_0, \infty}^{D_\mu}) \leq a_n(id_{p_0, \infty}^{D_\mu}) \lesssim 1$$

in combination with property (a) in Appendix A.

Let $2 \leq p_0 \leq \infty$. Because of $a_n(id_{p_0, \infty}^{D_\mu}) \asymp x_n(id_{p_0, \infty}^{D_\mu}) \asymp 1$ if $2n \leq D_\mu$ we may argue as in case of Weyl numbers, see the proof of Thm. 3.11 given above.

Now we consider the case $1 < p_0 < 2$ and $t > 1$. We define

$$id^* : s_{p_0, p_0}^{t, \Omega} b \rightarrow s_{\infty, \infty}^{0, \Omega} b \quad \text{and} \quad id_\mu^* : (s_{p_0, p_0}^{t, \Omega} b)_\mu \rightarrow (s_{\infty, \infty}^{0, \Omega} b)_\mu.$$

(7.6) and Lemma 7.6 yield

$$\|id_\mu^*\| \lesssim 2^{\mu(\frac{1}{p_0}-t)}$$

and

$$a_n(id_\mu^*) \asymp a_n(id_\mu^{**} : 2^{\mu(t-\frac{1}{p_0})} \ell_{p_0}^{D_\mu} \rightarrow \ell_\infty^{D_\mu}) \asymp 2^{\mu(-t+\frac{1}{p_0})} a_n(id_{p_0, \infty}^{D_\mu}) \quad (7.7)$$

for all $n \in \mathbb{N}$. Now we get as in Subsection 5.2, formula (6.7),

$$a_n(id^*) \lesssim \sum_{\mu=J+1}^L a_{n_\mu}(id_\mu^*) + 2^{L(-t+\frac{1}{p_0})}, \quad (7.8)$$

since $\rho = 1$ here. For

$$1 < \lambda < \frac{1}{2} + \frac{t}{2} \quad (7.9)$$

we define

$$n_\mu := D_\mu 2^{(J-\mu)\lambda}, \quad J+1 \leq \mu \leq L.$$

Then, as above,

$$n_\mu \leq \frac{D_\mu}{2} \quad \text{and} \quad \sum_{\mu=J+1}^L n_\mu \asymp J^{d-1} 2^J$$

follows. From (7.7) and $a_n(id_{p_0, \infty}^{D_\mu}) \asymp \min(1, D_\mu^{1-\frac{1}{p_0}} n^{-\frac{1}{2}})$ we conclude

$$\begin{aligned} a_{n_\mu}(id_\mu^*) &\lesssim 2^{\mu(-t+\frac{1}{p_0})} a_{n_\mu}(id_{p_0, \infty}^{D_\mu}) \\ &\lesssim 2^{\mu(-t+\frac{1}{p_0})} D_\mu^{1-\frac{1}{p_0}} (D_\mu \cdot 2^{(J-\mu)\lambda})^{-\frac{1}{2}} \\ &\lesssim 2^{\mu(-t+\frac{1}{2})} \mu^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} 2^{-\frac{1}{2}(J-\mu)\lambda}. \end{aligned}$$

This leads to

$$\begin{aligned} \sum_{\mu=J+1}^L a_{n_\mu}(id_\mu^*) &\lesssim \sum_{\mu=J+1}^L 2^{\mu(-t+\frac{1}{2})} \mu^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} 2^{-\frac{1}{2}(J-\mu)\lambda} \\ &\lesssim 2^{J(-t+\frac{1}{2})} J^{(d-1)(\frac{1}{2}-\frac{1}{p_0})}, \end{aligned}$$

since λ satisfies $\lambda < \frac{1}{2} + \frac{t}{2}$, see (7.9), guaranteeing the convergence of the series in that way. Now we choose L large enough such that

$$2^{L(-t+\frac{1}{p_0})} \lesssim 2^{J(-t+\frac{1}{2})} J^{(d-1)(\frac{1}{2}-\frac{1}{p_0})}.$$

In view of (7.8) this yields

$$a_n(id^*) \lesssim 2^{J(-t+\frac{1}{2})} J^{(d-1)(\frac{1}{2}-\frac{1}{p_0})}.$$

This proves the estimate from above.

Step 2. Let $p_0 = 1$. Then we use

$$a_n(id_{1, \infty}^{D_\mu}) \lesssim n^{-\frac{1}{2}}$$

if $2n \leq D_\mu$, see [62]. This is just the limiting case of Step 1. So we argue as there.

Step 3. It remains to consider the following case: $1 < p_0 < 2$, $s = 0$ and $\frac{1}{p_0} < t < 1$.

Substep 3.1. Estimate from above. In this case we define

$$L := \left\lceil J \frac{p'_0}{2} + (d-1)p'_0 \left(\frac{1}{p_0} - \frac{1}{2} \right) \log J \right\rceil \quad (7.10)$$

and

$$n_\mu := \left\lceil D_\mu 2^{(\mu-L)\beta+J-\mu} \right\rceil, \quad J+1 \leq \mu \leq L,$$

for some $\beta > 0$. Here p'_0 is the conjugate of p_0 , i.e., $\frac{1}{p_0} + \frac{1}{p'_0} = 1$. Again we have

$$n_\mu \leq \frac{D_\mu}{2} \quad \text{and} \quad \sum_{\mu=J+1}^L n_\mu \asymp J^{d-1} 2^J.$$

From (7.7), (7.8) and $a_n(id_{p_0, \infty}^{D_\mu}) \asymp \min(1, D_\mu^{1-\frac{1}{p_0}} n^{-\frac{1}{2}})$ we get

$$\begin{aligned} a_n(id^*) &\lesssim \sum_{\mu=J+1}^L 2^{\mu(-t+\frac{1}{p_0})} D_\mu^{1-\frac{1}{p_0}} [D_\mu 2^{(\mu-L)\beta+J-\mu}]^{-\frac{1}{2}} + 2^{L(-t+\frac{1}{p_0})} \\ &\lesssim \sum_{\mu=J+1}^L 2^{\mu(-t+1-\frac{\beta}{2})} 2^{\frac{L\beta-J}{2}} \mu^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} + 2^{L(-t+\frac{1}{p_0})}. \end{aligned}$$

The condition $t < 1$ guarantees that we can choose $\beta > 0$ such that $-t+1-\frac{\beta}{2} > 0$. Then we have

$$\begin{aligned} a_n(id^*) &\lesssim 2^{L(-t+1-\frac{\beta}{2})} 2^{\frac{L\beta-J}{2}} J^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} + 2^{L(-t+\frac{1}{p_0})} \\ &= 2^{L(-t+1)} 2^{-\frac{J}{2}} J^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} + 2^{L(-t+\frac{1}{p_0})}. \end{aligned}$$

Now, replacing L by the value in (7.10), a simple calculation yields

$$a_n(id^*) \lesssim 2^{J\frac{p'_0}{2}(-t+\frac{1}{p_0})} J^{(d-1)[\frac{p'_0}{2}(-t+\frac{1}{p_0})+t-\frac{1}{p_0}]}.$$

Rewriting this in dependence on n we obtain

$$a_n(id^*) \lesssim n^{-\frac{p'_0}{2}(t-\frac{1}{p_0})} (\log n)^{(d-1)(t-\frac{1}{p_0})}.$$

This proves the estimate from above.

Substep 3.2. Estimate from below. First of all, notice that we can prove $a_n(id^*) \geq a_n(id_\mu^*)$ as in (6.13). We choose $n = [D_\mu^{\frac{2}{p_0}}]$. By employing again (7.7) and

$$a_n(id_{p_0, \infty}^{D_\mu}) \asymp \min(1, D_\mu^{1-\frac{1}{p_0}} n^{-\frac{1}{2}})$$

we obtain the desired estimate. Finally, by making use of a lifting argument, see Lemma 7.3, and the counterpart of Lemma 7.1 we finish our proof. \blacksquare

7.5 Proof of interpolation properties of Weyl numbers

For the basics in interpolation theory we refer to the monographs [7, 30, 56].

To begin with we deal with Gelfand numbers. The n -th Gelfand number is defined as

$$c_n(T) = \inf_{M_n} \sup_{\|x\|_X \leq 1, x \in M_n} \|Tx\|_Y \quad (7.11)$$

where M_n is a subspace of X such that $\text{codim} M_n < n$, see also Section 4. Next we recall the interpolation properties of Gelfand numbers, for the case of Banach spaces we refer to Triebel [55].

Theorem 7.9. *Let $0 < \theta < 1$. Let X, Y, X_0, Y_0 be quasi-Banach spaces. Further we assume $Y_0 \cap Y_1 \hookrightarrow Y$ and the existence of a positive constant C with*

$$\|y\|_Y \leq C \|y\|_{Y_0}^{1-\theta} \|y\|_{Y_1}^\theta \quad \text{for all } y \in Y_0 \cap Y_1. \quad (7.12)$$

Then, if

$$T \in \mathcal{L}(X, Y_0) \cap \mathcal{L}(X, Y_1) \cap \mathcal{L}(X, Y)$$

it follows

$$c_{n+m-1}(T : X \rightarrow Y) \leq C c_n^{1-\theta}(T : X \rightarrow Y_0) c_m^\theta(T : X \rightarrow Y_1)$$

for all $n, m \in \mathbb{N}$. Here C is the same constant as in (7.12).

Proof. We follow the proof in [55]. Let L_n and L_m be subspaces of X such that $\text{codim} L_n < n$ and $\text{codim} L_m < m$ respectively. Then $\text{codim}(L_n \cap L_m) < m + n - 1$. Furthermore, by assumption, for all $x \in X$ we have $Tx \in Y_0 \cap Y_1$. From (7.11) and (7.12) we derive

$$\begin{aligned} c_{m+n-1}(T : X \rightarrow Y) &= \inf_{L_n, L_m} \sup_{\substack{\|x\|_X \leq 1 \\ x \in L_n \cap L_m}} \|Tx\|_Y \\ &\leq C \inf_{L_n, L_m} \sup_{\substack{\|x\|_X \leq 1 \\ x \in L_n \cap L_m}} \|Tx\|_{Y_0}^{1-\theta} \|Tx\|_{Y_1}^\theta \\ &\leq C \left(\inf_{L_n} \sup_{\substack{\|x\|_X \leq 1 \\ x \in L_n}} \|Tx\|_{Y_0} \right)^{1-\theta} \left(\inf_{L_m} \sup_{\substack{\|x\|_X \leq 1 \\ x \in L_m}} \|Tx\|_{Y_1} \right)^\theta \\ &= C c_n^{1-\theta}(T : X \rightarrow Y_0) c_m^\theta(T : X \rightarrow Y_1). \end{aligned}$$

The proof is complete. ■

Remark 7.10. Triebel [55] worked with Gelfand widths. For compact operators Gelfand widths and Gelfand numbers coincide, see also [55]. Hence, if we require

$$T \in \mathcal{K}(X, Y_0) \cap \mathcal{K}(X, Y_1) \cap \mathcal{L}(X, Y),$$

where $\mathcal{K}(X, Y)$ stands for the subspace of $\mathcal{L}(X, Y)$ formed by the compact operators, then Theorem 7.9 remains true for Gelfand widths. Without extra conditions on T Gelfand widths and Gelfand numbers may not coincide, see Edmunds and Lang [17] for a discussion of this question.

Now we ready prove the Thm. 4.2.

Proof of Theorem 4.2. Let $A \in \mathcal{L}(\ell_2, X)$ such that $\|A\| \leq 1$. Then from Thm. 7.9 we conclude

$$c_{n+m-1}(TA : \ell_2 \rightarrow Y) \leq C c_n^{1-\theta}(TA : \ell_2 \rightarrow Y_0) c_m^\theta(TA : \ell_2 \rightarrow Y_1).$$

Employing Remark 4.1(ii) we obtain

$$c_{n+m-1}(TA : \ell_2 \rightarrow Y) \leq C x_n^{1-\theta}(T : X \rightarrow Y_0) x_m^\theta(T : X \rightarrow Y_1).$$

Now taking the supremum with respect to A we find

$$x_{n+m-1}(T : X \rightarrow Y) \leq C x_n^{1-\theta}(T : X \rightarrow Y_0) x_m^\theta(T : X \rightarrow Y_1).$$

The proof is complete. ■

8 Appendix A - Weyl numbers of the embeddings $\ell_{p_0}^m \rightarrow \ell_p^m$

The Weyl numbers of $id : \ell_{p_0}^m \rightarrow \ell_p^m$ have been investigated at various places, we refer to Lubitz [29], König [27], Caetano [8, 9] and Zhang, Fang, Huang [65]. We shall need the following.

(a) ([29, Korollar 2.2] and [65]) Let $n, m \in \mathbb{N}$ and $2n \leq m$. Then we have

$$x_n(id_{p_0, p}^m) \asymp \begin{cases} 1 & \text{if } 2 \leq p_0 \leq p \leq \infty, \\ n^{\frac{1}{p} - \frac{1}{p_0}} & \text{if } 0 < p_0 \leq p \leq 2, \\ n^{\frac{1}{2} - \frac{1}{p_0}} & \text{if } 0 < p_0 \leq 2 \leq p \leq \infty, \\ m^{\frac{1}{p} - \frac{1}{p_0}} & \text{if } 0 < p < p_0 \leq 2. \end{cases}$$

(b) ([29, Korollare 2.6, 2.8, Satz 2.9]) Let $2 \leq p < p_0 \leq \infty$ and $n, m, k \in \mathbb{N}$, $k \geq 2$. Then we have

$$(i) \quad x_n(id_{p_0, p}^m) \lesssim \left(\frac{m}{n}\right)^{\frac{1}{r}} \text{ if } n \leq m, \quad \frac{1}{r} = \frac{1/p - 1/p_0}{1 - 2/p_0},$$

$$(ii) \quad x_n(id_{p_0, p}^m) \gtrsim m^{\frac{1}{p} - \frac{1}{p_0}} \text{ if } 1 \leq n \leq [m^{\frac{2}{p_0}}],$$

$$(iii) \quad x_n(id_{p_0, p}^{kn}) \asymp 1.$$

(c) ([65]) Let $0 < p \leq 2 < p_0 \leq \infty$ and $n, m \in \mathbb{N}$. Then

$$(i) \quad x_n(id_{p_0, p}^m) \gtrsim m^{\frac{1}{p} - \frac{1}{2}} \text{ if } n \leq \frac{m}{2},$$

$$(ii) \quad x_n(id_{p_0, p}^m) \gtrsim m^{\frac{1}{p} - \frac{1}{p_0}} \text{ if } n \leq m^{\frac{2}{p_0}}.$$

9 Appendix B - Function spaces of dominating mixed smoothness

9.1 Besov and Lizorkin-Triebel spaces on \mathbb{R}

Here we recall the definition and a few properties of Besov and Sobolev spaces defined on \mathbb{R} . We shall use the Fourier analytic approach, see e.g. [57]. Let $\varphi \in C_0^\infty(\mathbb{R})$ be a function such that $\varphi(t) = 1$ in an open set containing the origin. Then by means of

$$\varphi_0(t) = \varphi(t), \quad \varphi_j(t) = \varphi(2^{-j}t) - \varphi(2^{-j+1}t), \quad t \in \mathbb{R}, \quad j \in \mathbb{N}, \quad (9.1)$$

we get a smooth dyadic decomposition of unity, i.e.,

$$\sum_{j=0}^{\infty} \varphi_j(t) = 1 \quad \text{for all } t \in \mathbb{R},$$

and $\text{supp } \varphi_j$ is contained in the dyadic annulus $\{t \in \mathbb{R} : a 2^j \leq |t| \leq b 2^j\}$ with $0 < a < b < \infty$ independent of $j \in \mathbb{N}$.

Definition 9.1. Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$.

(i) The Besov space $B_{p,q}^s(\mathbb{R})$ is then the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ such that

$$\|f\|_{B_{p,q}^s(\mathbb{R})} := \left(\sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}[\varphi_j \mathcal{F}f](\cdot)\|_{L_p(\mathbb{R})}^q \right)^{1/q}$$

is finite.

(ii) Let $p < \infty$. The Lizorkin-Triebel space $F_{p,q}^s(\mathbb{R})$ is then the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ such that

$$\|f\|_{F_{p,q}^s(\mathbb{R})} := \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\mathcal{F}^{-1}[\varphi_j \mathcal{F}f](\cdot)|^q \right)^{1/q} \right\|_{L_p(\mathbb{R})}$$

is finite.

Remark 9.2. (i) There is an extensive literature about Besov and Lizorkin-Triebel spaces, we refer to the monographs [31], [57], [58] and [59]. These quasi-Banach spaces $B_{p,q}^s(\mathbb{R})$ and $F_{p,q}^s(\mathbb{R})$ can be characterized in various ways, e.g. by differences and derivatives, whenever s is sufficiently large, i.e., $s > \max(0, 1/p - 1)$ in case of Besov spaces and $s > \max(0, 1/p - 1, 1/q - 1)$ in case of Lizorkin-Triebel spaces. We refer to [57] for details.
(ii) The spaces $B_{p,q}^s(\mathbb{R})$ and $F_{p,q}^s(\mathbb{R})$ do not coincide as sets except the case $p = q$.

9.2 Besov and Lizorkin-Triebel spaces of dominating mixed smoothness

Detailed treatments of Besov and Lizorkin-Triebel spaces of dominating mixed smoothness are given at various places, we refer to the monographs [1, 47], the survey [46] as well as to the booklet [61].

If φ_j , $j \in \mathbb{N}_0$, is a smooth dyadic decomposition of unity as in (9.1), then by means of

$$\varphi_{\bar{j}} := \varphi_{j_1} \otimes \dots \otimes \varphi_{j_d}, \quad \bar{j} = (j_1, \dots, j_d) \in \mathbb{N}_0^d, \quad (9.2)$$

we obtain a smooth decomposition of unity on \mathbb{R}^d .

Definition 9.3. Let $0 < p, q \leq \infty$ and $t \in \mathbb{R}$.

(i) The Besov space of dominating mixed smoothness $S_{p,q}^t B(\mathbb{R}^d)$ is the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{S_{p,q}^t B(\mathbb{R}^d)} := \left(\sum_{\bar{j} \in \mathbb{N}_0^d} 2^{|\bar{j}|_1 t q} \|\mathcal{F}^{-1}[\varphi_{\bar{j}} \mathcal{F}f](\cdot)\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q}$$

is finite.

(ii) Let $0 < p < \infty$. The Lizorkin-Triebel space of dominating mixed smoothness $S_{p,q}^t F(\mathbb{R}^d)$ is the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{S_{p,q}^t F(\mathbb{R}^d)} := \left\| \left(\sum_{\bar{j} \in \mathbb{N}_0^d} 2^{|\bar{j}|_1 t q} |\mathcal{F}^{-1}[\varphi_{\bar{j}} \mathcal{F}f](\cdot)|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}$$

is finite.

Remark 9.4. (i) The most interesting property of these classes for us consists in the following: if

$$f(x) = \prod_{j=1}^d f_j(x_j), \quad x = (x_1, \dots, x_d), \quad f_j \in A_{p,q}^t(\mathbb{R}), \quad j = 1, \dots, d,$$

then $f \in S_{p,q}^t A(\mathbb{R}^d)$ and

$$\|f\|_{S_{p,q}^t A(\mathbb{R}^d)} = \prod_{j=1}^d \|f_j\|_{A_{p,q}^s(\mathbb{R})}, \quad A \in \{B, F\}.$$

I.e., Lizorkin-Triebel and Besov spaces of dominating mixed smoothness have a cross-quasi-norm.

(ii) These classes $S_{p,q}^t B(\mathbb{R}^d)$ as well as $S_{p,q}^t F(\mathbb{R}^d)$ are quasi-Banach spaces. If either $t > \max(0, (1/p) - 1)$ (B-case) or $t > \max(0, 1/p - 1, 1/q - 1)$ (F-case), then they can be characterized by differences, we refer to [47] and [60] for details.

(iii) Again the spaces $S_{p,q}^t B(\mathbb{R}^d)$ and $S_{p,q}^t F(\mathbb{R}^d)$ do not coincide as sets except the case $p = q$.

(iv) For $d = 1$ we have

$$S_{p,q}^t A(\mathbb{R}) = A_{p,q}^t(\mathbb{R}), \quad A \in \{B, F\}.$$

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